CENTRAL CHARGES, SYMPLECTIC FORMS, AND HYPERGEOMETRIC SERIES IN LOCAL MIRROR SYMMETRY

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Abstract. We study a cohomology-valued hypergeometric series which naturally arises in the description of (local) mirror symmetry. We identify it as a central charge formula for BPS states and study its monodromy property from the viewpoint of Kontsevich's homological mirror symmetry. In the case of local mirror symmetry, we will identify a symplectic form, and will conjecture an integral and symplectic monodromy property of a relevant hypergeometric series of Gel'fand-Kapranov-Zelevinski type.

Table of Contents

1. Introduction

Let us consider a (famous) hypergeometric series of one variable $[1, x] \in \mathbf{P}^1$;

$$w(x) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} x^n \tag{1.1}$$

This is a hypergeometric series of type ${}_4F_3(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; 1, 1, 1; x)$ which arises in the mirror symmetry of quintic hypersurface $X_5 \subset \mathbf{P}^4$. This hypergeometric series (1.1) represents one of the period integrals of the mirror quintic $X_5^\vee$ and satisfies

the following differential equation (Picard-Fuchs) equation:

$$\{\theta_x^4 - 5^5 x(\theta_x + \frac{4}{5})(\theta_x + \frac{3}{5})(\theta_x + \frac{2}{5})(\theta_x + \frac{1}{5})\} w(x) = 0 \; , \tag{1.2}$$

where $\theta_x := x\frac{d}{dx}$. See the original work by Candelas et al [CdOGP] for the description of the mirror family and its period integrals.

As it is clear in the form of differential equation, the regular singularity at $x = 0$ has a distinguished property, i.e., the monodromy around this point is *maximally unipotent* [Mor]. In physics, the point $x = 0$ is called a *large complex structure limit* and plays important roles, e.g., near this point, we evaluate the quantum corrections to the classical geometry of the σ-model on the quintic X_5. Here we focus on the construction of local solutions about $x = 0$ by the classical Frobenius method;

$$w_0(x) := w(x) \; , \; w_1(x) := \tfrac{\partial}{\partial\rho} w(x;\rho)|_{\rho=0} \; ,$$
$$w_2(x) := \tfrac{\partial^2}{\partial\rho^2} w(x;\rho)|_{\rho=0} \; , \; w_3(x) := \tfrac{\partial^3}{\partial\rho^3} w(x;\rho)|_{\rho=0} \; ,$$

where $w(x;\rho) := \sum_{n\geq 0} \frac{\Gamma(1+5(n+\rho))}{\Gamma(1+(n+\rho))^5} x^{n+\rho}$. With the mirror symmetry of X_5 in $\mathbf{P}^4$ and $X_5^\vee$ in mind, we introduce the following cohomology-valued hypergeometric series;

$$w(x;\frac{J}{2\pi i}) := w(x) + w_1(x)(\frac{J}{2\pi i}) + w_2(x)(\frac{J}{2\pi i})^2 + w_3(x)(\frac{J}{2\pi i})^3 \; , \tag{1.3}$$

where J is the ample, integral generator of $\mathrm{Pic}(X_5) = H^{1,1}(X_5) \cap H^2(X_5, \mathbf{Z})$. In this form, we note that the classical Frobenius method is concisely summarized as the Taylor expansion $w(x;\rho)|_{\frac{J}{2\pi i}}$ with respect to the nilpotent element J. Although this seems just an advantage in bookkeeping, the following observation reported in [Hos] indicates that we have *more than* that in (1.3):

Observation: *Arrange the Taylor series expansion of the cohomology-valued hypergeometric series* $w(x;\frac{J}{2\pi i})$ *as*

$$w(x;\frac{J}{2\pi i}) = w^{(0)}(x) + w^{(1)}(x)(J - \frac{c_2(X_5)J}{12} - \frac{11}{2}\frac{J^2}{5}) + w^{(2)}(x)\frac{J^2}{5} + w^{(3)}(x)(-\frac{J^3}{5}). \tag{1.4}$$

Then the monodromy matrices of the coefficient hypergeometric series $w^{(0)}(x)$*,* $w^{(1)}(x)$*,* $w^{(2)}(x)$*,* $w^{(3)}(x)$ *are integral and symplectic (with respect to the symplectic form of the mirror, see below).*

The integral and symplectic properties of the solutions $w^{(k)}(x)$ $(k = 0, 1, 2, 3)$ stem from those of the middle homology group of the mirror $H_3(X_5^\vee, \mathbf{Z})$. The point here is that we can recover these integral, symplectic properties from the series $w(x;\frac{J}{2\pi i})$ through suitably arranging a basis of $H^{even}(X, \mathbf{Q})$ near the large complex structure limit. Since there is a symplectic, integral structure on $H^{even}(X, \mathbf{Q})$ which comes from $K(X)$, the Grothendieck group of algebraic vector bundles on X, it is natural to conjecture (Conjecture 2.2) that the cohomology-valued hypergeometric series describes a "pairing" between $K(X)$ and $H_3(X_5^\vee, \mathbf{Z})$, as proposed in [Hos].

The aims (and main results) of this paper are, 1) to make an interpretation of the cohomology-valued hypergeometric series from the viewpoint of homological mirror symmetry, and to give a general definition of the so-called central charge of BPS states (**Conjecture 2.2**, **Conjecture 6.3**), 2) to present supporting evidences for

the interpretation in the cases from local mirror symmetry in dimensions two and three (Section 5, Section 6).

The idea of the local mirror symmetry is simply to focus on the Calabi-Yau geometry near rigid curves or (Fano) surfaces in a Calabi-Yau manifold [CKYZ]. Recently this idea has been providing us useful testing grounds for several duality symmetries proposed in string theory. In particular, when the relevant Calabi-Yau geometry is toric, remarkable progresses have been made (, see [AKMV] and references therein). In this paper, for the local Calabi-Yau geometries, we will consider the crepant resolutions $\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ of the two dimensional canonical singularity and also $\widehat{\mathbf{C}^3/G}$ with finite abelian group $G \subset SL(3,\mathbf{C})$. Although we restrict our attentions to these cases, our arguments should extend to more general local Calabi-Yau geometries.

When studying relevant Gel'fand-Kapranov-Zelevinski(GKZ) hypergeometric series, we will find its connection to the primitive forms introduced by K.Saito in the deformation theory of singularities (**Proposition 4.2**, **Proposition 4.4**). This seems to be interesting in its own right, since GKZ hypergeometric series may provide a simple way to express the 'period integrals' of the primitive forms (or oscillating integrals) in the theory of singularities.

The construction of this paper is as follows: In section 2, we give a definition of the central charge assuming Kontsevich's homological mirror symmetry. Following the observations made in [Hos], we interpret the cohomology-valued hypergeometric series as the central charge for the cases of X compact. In section 3, we consider the local mirror symmetry of $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ and study in detail the GKZ hypergeometric series in this case. In section 4, we connect the GKZ system to K.Saito's system of differential equations which describes the deformation of singularities. In section 5, we extract a (homological) mirror map $mir : K^c(X) \xrightarrow{\sim} H_2(Y,\mathbf{Z})$ from our interpretation of the cohomology-valued hypergeometric series, and find a consistency with the mirror symmetry in terms of the hyperkähler rotation. In section 6, we formulate our conjecture on cohomology-valued hypergeometric series for three dimensional cases $X = \widehat{\mathbf{C}^3/G}$. We also find a closed formula for the prepotential. Two explicit examples are presented to support our conjecture. In Appendix A and B, we evaluate period integrals over vanishing cycles for the mirror geometry of $X = \widehat{\mathbf{C}^3/\mathbf{Z}_3}$. There, we see that a certain noncompact vanishing cycle arises in the description of the isomorphism $mir : K^c(X) \xrightarrow{\sim} H_3(Y,\mathbf{Z})$. The relation to the period integral of a primitive form is also elucidated.

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2. Central charge formula in terms of $w(x; \frac{J}{2\pi i})$

Here, we will interpret the cohomology-valued hypergeometric series from the viewpoint of Kontsevich's homological mirror symmetry [Ko].

Let X be a Calabi–Yau 3 fold and Y be a mirror of X. Following Kontsevich, we consider the bounded derived category $D^b(Coh(X))$ of coherent sheaves (D-branes of B type) on X. On the other hand, for the mirror side, we consider the derived Fukaya category $DFuk(Y,\beta)$ with the Kähler form viewed as a symplectic form β. The objects of the latter category consist of (graded) Lagrangian submanifolds with a flat $U(1)$ bundle on each of them (D-branes of type A) and morphisms are given by the Floer homology for Lagrangian submanifolds, and this category forms a triangulated category (see [FO3] for a more precise definition). Kontsevich proposed that these two different categories are equivalent (as triangulated categories) when X and Y are mirror symmetric, and also that this should be a mathematical definition of mirror symmetry. This conjecture itself is of great interest; however let us consider this conjecture at a more tractable level, i.e. at the level of cohomology or K-groups as shown in the second line below;

$$
\begin{array}{ccc}
D^b(Coh(X)) & \xrightarrow{Mir} & DFuk(Y,\beta) \\
\downarrow & & \downarrow \\
K(X) \text{ or } H^{even}(X,\mathbf{Q}) & \xrightarrow{mir} & H_3(Y,\mathbf{Z})
\end{array}
\tag{2.1}
$$

where the left vertical arrow represents the natural map from $D^b(Coh(X))$ to the K-group of algebraic vector bundles $K(X)$, and its composition with the Chern character homomorphism $ch(\cdot)$ if we further map to $H^{even}(X,\mathbf{Q}) = \oplus_{p=0}^{3} H^{2p}(X,\mathbf{Q})$. The right vertical arrow is given simply by taking the homology classes of the graded Lagrangian cycles. In the second line, the equivalence, Mir, of the two categories becomes simply an isomorphism, mir, between the K-group and $H_3(Y,\mathbf{Z})$. We should note that this is not simply an isomorphism but an isomorphism with the symplectic structures, i.e.

$$
mir:\ (K(X), \chi(E,F)) \ \overset{\sim}{\to}\ (H_3(Y,\mathbf{Z}), \#(L_E \cap L_F)),
$$

where $\chi(E,F) = \int_X ch(E^\vee) ch(F) Todd_X$ and $\#(L_E \cap L_F) := \int_Y \mu_{L_E} \cup \mu_{L_F}$ with the Poincaré duals $\mu_{L_E}, \mu_{L_F} \in H^3(Y,\mathbf{Z})$ of the mirror homology cycles $L_E := mir(E), L_F := mir(F)$. Here we remark that the Euler number $\chi(E,F)$ is anti-symmetric due to Serre duality and $K_X = 0$, and also non-degenerate. Thus $\chi(E,F)$ introduces a symplectic structure on $K(X)$, which is the mirror of the symplectic structure on $H_3(Y,\mathbf{Z})$.

In the diagram (2.1), we assume that a complex structure is fixed to define $D^b(Coh(X))$. Correspondingly the symplectic form (Kähler form) β, determined by the mirror map, is fixed in the right hand side. On the other hand, we may change the (complexified) Kähler class of X which corresponds to the complex structure moduli of Y under the mirror map. Changing the (complexified) Kähler structure amounts to changing the polarization and thus results in varying the stability condition on the sheaves on X. This change of the stability (Π-stability) condition has been studied in [Do] as a stability of BPS D-branes and its mathematical aspects are elaborated in [Br]. Here, we will not go into the detailed definition of Π-stability, but we will propose a closed formula for the *central charge* which is indispensable for an explicit description of Π-stability.

Definition 2.1. (Central charge formula.) Assume $K(X)$ is torsion-free, and let $E_1, \cdots, E_r$ be a $\mathbf{Z}$-basis of $K(X)$. Let $\Omega(Y_x)$ be a holomorphic 3-form of the mirror family $\{Y_x\}_{x \in \mathcal{B}}$ of X with $x = (x_1, \cdots, x_r)$ being the local parameter near a large complex structure limit. Denote by $\mathbf{C}\{x\}[\log x]$ the polynomial ring of $\log x_1, \cdots, \log x_r$ over the ring $\mathbf{C}\{x\}$ of convergent power series. Under the mirror symmetry (2.1), we define the following $\mathcal{Z}_x$ as an element in $K(X) \otimes \mathbf{C}\{x\}[\log x]$:

$$\mathcal{Z}_x := \sum_{i,j} \int_{mir(E_i)} \Omega(Y_x)\, \chi^{ij}\, E_j^{\vee} \tag{2.2}$$

with $(\chi^{ij}) := (\chi(E_i, E_j))^{-1}$. Then the central charge of $F \in K(X)$ is defined by

$$Z_t(F) = \int_X ch(F)\, ch(\mathcal{Z}_x)\, Todd_X \quad , \tag{2.3}$$

where $t = t(x)$ (or $x = x(t)$) is the mirror map connecting the local parameters $x_1, \cdots, x_r$ to those $t_1, \cdots, t_r$ of the (complexified) Kähler moduli space of X.

In the above definition, it should be noted that $\mathcal{Z}_x$ does not depend on the choice of a basis $E_1, \cdots, E_r$. Also the central charge $Z_t(F)$ contains full 'quantum corrections' as a function of $t_1, \cdots, t_r$ (cf. the asymptotic formula given in [Do]).

Let us connect our hypergeometric series $w(x; \frac{J}{2\pi i})$ to the central charge above. Before doing this, we remark that, in the mirror symmetry of Calabi–Yau hypersurfaces due to Batyrev[Ba1], the hypergeometric series (1.1) is naturally generalized to the Gel'fand–Kapranov–Zelevinski (GKZ) hypergeometric series of multi-variables $x_1, \cdots, x_r$ [GKZ1],[Ba2]. Using the GKZ hypergeometric series, and also suitable integral, (semi-)ample generators $J_1, \cdots, J_r$ of $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$, we have a cohomology-valued hypergeometric series $w(x; \frac{J}{2\pi i})$ as a generalization of (1.3) (see Sect.2 of [Hos] for more details).

Conjecture 2.2. *The cohomology-valued hypergeometric series (1.3) gives the expression $ch(\mathcal{Z}_x) \in H^{even}(X) \otimes \mathbf{C}\{x\}[\log x]$;*

$$w(x_1, \cdots, x_r; \frac{J_1}{2\pi i}, \cdots, \frac{J_r}{2\pi i}) = \sum_{i,j} \int_{mir(E_i)} \Omega(Y_x)\, \chi^{ij}\, ch(E_j^{\vee}) \ . \tag{2.4}$$

Using this, and also the mirror map $t = t(x)$, we can write the central charge $Z_t(F)$ of $F \in K(X)$ as

$$Z_t(F) = \int_X ch(F)\, w(x; \frac{J}{2\pi i})\, Todd_X \quad . \tag{2.5}$$

Here we note that the hypergeometric series has a finite radius of convergence and shows a monodromy property when it is analytically continued around its (regular) singularities. As noticed by Kontsevich, this monodromy property should be mirrored to some linear (symplectic) transformations on $ch(E_i)$ which come from Fourier–Mukai transforms on $D^b(Coh(X))$. If we postulate that the cohomology-valued hypergeometric series has an invariant meaning under these monodromy actions, our cohomology-valued hypergeometric series $w(x; \frac{J}{2\pi i})$ provides a connection between these two different 'monodromy' transforms on the two sides. The conjectural formula (2.4) has been tested in the case X is an elliptic curve, (lattice polarized) K3 surfaces, and several Calabi–Yau hypersurfaces in [Hos]. Cohomology-valued hypergeometric series are utilized also in [Gi],[LLY],[Sti] in a slightly different form. We remark that, for our Conjecture 2.2 to work, the definition given in [Hos] is crucial.

As studied in [Mu] for the cases of K3 surfaces and abelian varieties, and in [Or] for general, the Fourier–Mukai transform is a self-equivalence of the category $D^b(Coh(X))$ which takes the form

$$\Phi^{\mathcal{P}}(\cdot) = \mathbf{R}p_{2*}(p_1^*(\cdot) \overset{\mathbf{L}}{\otimes} \mathcal{P})$$

where $\mathcal{P}$ is an object in $D^b(Coh(X \times X))$, called the *kernel*, and p_1 and p_2 are, respectively, the natural projections to the first and the second factor from $X \times X$. Due to a result in [Or], we may always assume the above form, i.e., there exists a suitable kernel $\mathcal{P}$, for any equivalence $\Phi : D^b(Coh(X)) \simeq D^b(Coh(X))$ as triangulated category. It is rather easy to see that the monodromy transforms around the large complex structure limit are given by tensoring invertible sheaves, which may be expressed by the kernels;

$$\mathcal{P} : \ \cdots \to 0 \to \mathcal{O}_\Delta \times p_2^*(\mathcal{O}_X(D)) \to 0 \cdots \quad ,$$

with $D \in Pic(X)$ and Δ representing the diagonal in $X \times X$. Kontsevich predicted that a monodromy transform associated to a vanishing cycle, the Picard–Lefschetz transform, has its mirror FM transform with its kernel,

$$\mathcal{P} : \ \cdots \to 0 \to \mathcal{O}_{X \times X} \to \mathcal{O}_X \to 0 \cdots \quad .$$

Seidel and Thomas [ST] (and Horja [Hor]) generalized the above kernel, associating it to so-called spherical objects $\mathcal{E} \in D^b(Coh(X))$ with the defining property: $Ext^i(\mathcal{E}, \mathcal{E}) = 0$ $(i \neq 0, n)$, $= \mathbf{C}$ $(i = 0, n)$ where $n = \dim X$. For each spherical object, we have a kernel given by the mapping cone;

$$\mathcal{P} = Cone(\mathcal{E}^\vee \otimes^{\mathbf{L}} \mathcal{E} \to \mathcal{O}_\Delta) \quad .$$

The equivalence $\Phi^{\mathcal{P}}$ is called a Seidel-Thomas twist. We will see these equivalences in the corresponding monodromy property of our hypergeometric series.

3. Local mirror symmetry I — $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$

In this section and the subsequent sections, we will test our Conjecture 2.2 for the case of mirror symmetry of non-compact toric Calabi-Yau manifolds (local mirror symmetry). Batyrev's idea of mirror symmetry [Ba1] still makes sense for such non-compact toric Calabi-Yau manifolds as well as the attractive proposal by Strominger-Yau-Zaslow(SYZ)[SYZ], which is closely related to the homological mirror symmetry (2.1). Mirror symmetry of non-compact toric Calabi-Yau manifolds and also Fano varieties are formulated in a language of Landau-Ginzburg theory in [HIV].

(3-1) Mirror symmetry and hyperkähler rotation. Let us consider the minimal resolution of a two dimensional simple singularity; $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$. This is an example of two dimensional, non-compact, toric Calabi-Yau manifold. Two dimensional Calabi-Yau manifolds are hyperkähler, and it is known that the mirror symmetry of them is well-understood by the hyperkähler rotation, see e.g. [GW][Huy]. Our minimal resolution X has a natural hyperkähler structure, and therefore its mirror is X itself with a different complex structure after a suitable rotation. To describe the mirror symmetry, let us first write the quotient $\mathbf{C}/\mathbf{Z}_{\mu+1}$ by a hypersurface $UV = W^{\mu+1}$ in $\mathbf{C}^3$. Bowing up the singularities at the origin μ times results in the minimal resolution X, and thereby we introduce exceptional curves $C_i \cong \mathbf{P}^1$ $(i = 1, \cdots, \mu)$. On the other hand, we may deform the defining

equation $UV = W^{\mu+1}$ to $UV = a_1 + a_2 W + \cdots + a_{\mu+2} W^{\mu+1}$ with introducing finite sizes to vanishing cycles $L_i \approx S^2$ $(\mu = 1, \cdots, \mu)$. Note that the number of vanishing cycles are given by the Milnor number $\mu = \dim R_J$, where R_J is the Jacobian ring of the singularity $UV = W^{\mu+1}$. The vanishing cycles are Lagrangians, and become holomorphic cycles under a suitable hyperkähler rotation. The holomorphic geometry after the rotation is bi-holomorphic to the blown-up geometry of X. If we forget about the role of the B-fields, this describes the mirror symmetry of X. (See e.g. [Huy] and references therein for full details of the mirror symmetry via the hyperkähler rotation.) Here we note that the intersection form of these vanishing cycles are given in both holomorphic and Lagrangian geometry by

$$(C_i \cdot C_j) = (\# L_i \cap L_j) = -\mathcal{C}_{ij} \quad ,$$

where $\mathcal{C}_{ij}$ $(1 \le i, j \le \mu)$ is the Cartan matrix for the root system of A_μ.

(3-2) GKZ hypergeometric series. The minimal resolution $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ is a (non-compact) toric variety whose resolution is described by a two dimensional fan Σ with its integral generators ν_i for one dimensional cones (see Fig.1). We set

$$A = \{\nu_1, \nu_2, \cdots, \nu_{\mu+2}\} = \{\ (1,0), (1,1), \cdots, (1, \mu+1)\ \} \ .$$

The half-lines $\overline{o\nu_i}$ $(i = 1, \cdots, \mu+2)$ from the origin $o = (0,0)$ constitute the one dimensional cones of the resolution diagram Σ. In Batyrev's mirror symmetry, the resolution diagram of X, up to flop operations, is identified with the Newton polytope of the defining equation of its mirror Y. This construction has been extended to the cases of non-compact toric Calabi-Yau manifiods in [CKYZ][HIV], and for $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$, the mirror Y is given by $U^2 + V^2 + f_\Sigma(W) = 0 \subset \mathbf{C}^2 \times \mathbf{C}^*$ with

$$f_\Sigma(W) = a_1 + a_2 W^1 + a_3 W^2 + \cdots + a_{\mu+2} W^{\mu+1} \ ,$$

where $(U, V, W) \in \mathbf{C}^2 \times \mathbf{C}^*$. In the case of local mirror symmetry, the meaning of the period integrals of holomorphic two form becomes less clear than the compact cases. Here, motivated by [CKYZ], we consider the following integral

$$\Pi_\gamma(a) := \frac{i}{2\pi^2} \int_\gamma \frac{1}{U^2 + V^2 + f_\Sigma(a; W)} dU dV \frac{dW}{W} \quad , \tag{3.1}$$

for each cycle in the complement of the hypersurface

$$\gamma \in H_3(\mathbf{C}^2 \times \mathbf{C}^* \setminus Y, \mathbf{Z}).$$

Note that we have an isomorphism $H_3(\mathbf{C}^2 \times \mathbf{C}^* \setminus Y, \mathbf{Z}) \simeq H_2(Y, \mathbf{Z})$ for the hypersurface $Y \subset \mathbf{C}^2 \times \mathbf{C}^*$ (see P.46 of [Di] for example).

In the next section, we will relate this 'period integral' to K. Saito's primitive form (or Gel'fand-Leray form) in the deformation theory of singularities [Sa][AGV]. Here we set up a hypergeometric differential equations (GKZ system) satisfied by $\Pi_\gamma(a)$. This GKZ hypergeometric system is also referred to as A-hypergeometric system since it is described by the set A through the *lattice of relations*,

$$\mathcal{L} = \{\ (l_1, l_2, \cdots, l_{\mu+2}) \,|\, l_1\nu_1 + l_2\nu_2 + \cdots + l_{\mu+2}\nu_{\mu+2} = (0,0)\ \}. \tag{3.2}$$

Using this lattice of relations, the system is written as

$$\Box_l \Pi_\gamma(a) = 0 \ (l \in \mathcal{L}) \ \ , \ \ \mathcal{Z}_i \Pi_\gamma(a) = 0 \ (i = 1, 2) \ , \tag{3.3}$$

where

$$\Box_l = \left(\frac{\partial}{\partial a}\right)^{l^+} - \left(\frac{\partial}{\partial a}\right)^{l^-} \quad , \quad \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{pmatrix} = \begin{pmatrix} \theta_1 + \theta_2 + \cdots + \theta_{\mu+2} \\ \theta_2 + 2\theta_3 \cdots + (\mu+1)\theta_{\mu+2} \end{pmatrix}$$

with $l = l^+ - l^-$ ($l_i^\pm := \frac{1}{2}(l_i \pm |l_i|)$ and $\theta_k = a_k \frac{\partial}{\partial a_k}$. From the formal solutions of this system in [GKZ1], it is easy to write down our $w(x; \frac{J}{2\pi i})$ (, see (3-3) below). An important aspect of this system is that there is a natural toric compactification $\mathcal{M}_{Sec(\Sigma)}$ of the parameter space $\{(a_1, \cdots, a_{\mu+2}) \in (\mathbf{C}^*)^{\mu+2}/(\mathbf{C}^*)^2\}$ in terms of the *secondary fan* $Sec(\Sigma)$ (, see [GKZ2] and references therein), where the quotient by $(\mathbf{C}^*)^2$ is represented by the (scaling) operators $\mathcal{Z}_1$ and $\mathcal{Z}_2$. This compactification plays an important role in the applications of mirror symmetry to Gromov-Witten invariants, since the large radius limit appears as an intersection point of the boundary toric divisors. Connecting the GKZ system to K. Saito's differential equations in singularity theory, we will see in section 4 that this compactification will also provide a natural way to compactify the deformation space of singularities which is local in nature (, see (4-1) for a brief description of the deformation space).

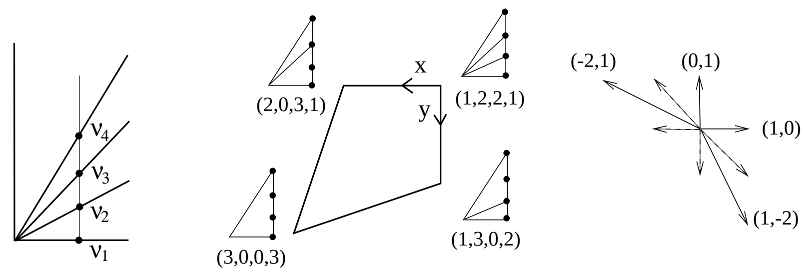

Fig.1. *The resolution diagram (left), the secondary polytope (middle), and the secondary fan $Sec(\Sigma)$ (right) for $\mu = 2$. The secondary polytope has its vertices parametrized by (regular) triangulations of the polytope as shown. For a triangulation T, the corresponding vertex is determined by a vector $v_T = (\varphi_T(\nu_1), \varphi_T(\nu_2), \varphi_T(\nu_3), \varphi_T(\nu_4))$ with $\varphi_T(\nu_i) = \sum_{\nu_i \prec \sigma} Vol(\sigma)$. As we see in this example ($\mu = 2$), the convex hull of these vertices lies on $\mathcal{L}_{\mathbf{R}} + v_T$ with a vector v_T, where $\mathcal{L}_{\mathbf{R}} = \mathcal{L} \otimes \mathbf{R}$. Normal cones of the secondary polytope determines the secondary fan.*

(3-3) Example ($\mu = 2$). Here we present some explicit calculations of cohomology-valued hypergeometric series for the case $\mu = 2$, instead of giving general formulas valid for any μ. In this case, the cohomology-valued hypergeometric series is simple and takes the following form;

$$w(x; y; \frac{J_1}{2\pi i}, \frac{J_2}{2\pi i}) = w(x; y; \rho_1, \rho_2)|_{\rho_1 = \frac{J_1}{2\pi i}, \rho_2 = \frac{J_2}{2\pi i}} ,$$

where $w(x; y; \rho_1, \rho_2) = \sum_{n,m \geq 0} c(n + \rho_1, m + \rho_2) x^{n+\rho_1} y^{m+\rho_2}$ with

$$c(n, m) = 1/\big(\Gamma(1 + n)\Gamma(1 - 2n + m)\Gamma(1 + n - 2m)\Gamma(1 + m)\big) .$$

J_1 and J_2 are semi-ample classes which are dual to the exceptional curves C_1 and C_2 (, i.e. the toric divisors D_{ν_2} and D_{ν_3}), respectively. The local parameters $x := \frac{a_1 a_3}{a_2^2}, y := \frac{a_2 a_4}{a_3^2}$ are depicted in Fig.1. Here we remark that the secondary polytope in Fig.1 sits on a translation of $\mathcal{L}_{\mathbf{R}}$, i.e. $\mathcal{L}_{\mathbf{R}} + v_T$, and the summation in $\sum_{n,m \geq 0} c(m, m) x^m y^n$ is in fact that over the integral points inside the dual of the normal cone from the vertex $v_T = (1, 2, 2, 1)$. We may recognize this fact in the relation $x^n y^m = a^{nl^{(1)} + ml^{(2)}}$, where by definition $l^{(1)} = (1, -2, 1, 0)$ and $l^{(2)} = (0, 1, -2, 1)$ generate the dual of the normal cone form the vertex v_T. We remark

also that the coefficient $c(n,m)$ are determined from the entries of $nl^{(1)}+ml^{(2)}$;

$$(n\ m)\begin{pmatrix}1 & -2 & 1 & 0\\ 0 & 1 & -2 & 1\end{pmatrix}=\begin{pmatrix}n & -2n & n & 0\\ 0 & m & -2m & m\end{pmatrix}\ ,$$

where the row vectors determine the arguments of the gamma functions in the denominator of $c(n,m)$. In this form, it is rather clear how the formula generalizes to arbitrary μ (, see [CKYZ],[dOFS] for example).

Since the ring $H^{even}(X,\mathbf{Z})$ is generated by $1, J_1, J_2$, we have the expansion;

$$w(\vec{x},\frac{\vec{J}}{2\pi i})=1+w_1(x,y)J_1+w_2(x,y)J_2\ \ ,$$

with $w_1(x,y)=\frac{1}{2\pi i}\log x+\cdots$, $w_2(x,y)=\frac{1}{2\pi i}\log y+\cdots$. The mirror map is defined from the relations,

$$q_1:=e^{2\pi i w_1(x,y)}=x(1+g_1(x,y))\ \ ,\ \ q_2:=e^{2\pi i w_2(x,y)}=y(1+g_2(x,y))\ \ , \tag{3.4}$$

where $g_1(x,y), g_2(x,y)$ represent powerseries of x and y. Then $t_1:=\frac{1}{2\pi i}\log q_1(=w_1(x,y))$ and $t_2:=\frac{1}{2\pi i}\log q_2(=w_2(x,y))$ are the complexified Kähler moduli and measure the volumes of the exceptional curves C_1, and C_2, respectively. The inverse relation $x=x(q_1,q_2), y=y(q_1,q_2)$ of (3.4) is often referred to as the mirror map, and has the following properties (see Proposition 4.4 for a proof);

Proposition 3.1.

1) *The mirror map $x=x(q_1,q_2), y=y(q_1,q_2)$ is rational and is expressed by $x=\frac{a_1a_3}{a_2^2}, y=\frac{a_2a_4}{a_3^2}$ with a_i's determined through*

$$a_1+a_2W+a_3W^2+a_4W^3=(1+W)(1+q_1W)(1+q_1q_2W)\ \ . \tag{3.5}$$

Concretely, it has the form;

$$x=\frac{q_1(1+q_2+q_1q_2)}{(1+q_1+q_1q_2)^2}\ ,\ y=\frac{q_2(1+q_1+q_1q_2)}{(1+q_1+q_1q_2)^2}\ \ . \tag{3.6}$$

2) *The discriminant of the GKZ system (3.3) consists of three components; $x=0, y=0$ and $dis(x,y)=0$ with*

$$dis(x,y)=1-4x-4y+18xy-27x^2y^2=\frac{(1-q_1)^2(1-q_2)^2(1-q_1q_2)^2}{(1+q_1+q_1q_2)^2(1+q_2+q_1q_2)^2}\ .$$

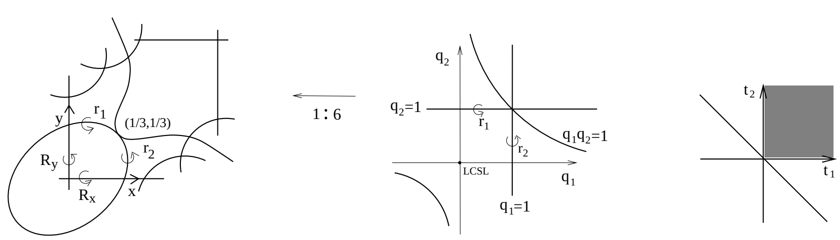

Fig.2. *The discriminant $dis(x,y)=0$ in $\hat{\mathcal{M}}_{Sec(\Sigma)}$ (left), the mirror map (3.6) from the q_1q_2-plane to the xy-plane (middle), and the complexified Kähler moduli t_1,t_2 with the complexified Kähler cone (right). The discriminant is an elliptic curve with a node at $(x,y)=(\frac{1}{3},\frac{1}{3})$, which corresponds to $(q_1,q_2)=(1,1)$. The mirror map is $6:1$ at generic (q_1,q_2). Over the discriminant, it is $3:1$ and the inverse image of $dis(x,y)=0$ consists of the three lines $q_1=1, q_2=1, q_1q_2=1$ in the q-plane.*

It is easy to see that $x=0, y=0$ are the toric boundary divisors whose intersection point define the large complex structure. Over the zeros of the discriminant $dis(x,y)$, we see vanishing cycles in $f_\Sigma(a,W)+U^2+V^2=0 \subset \mathbf{C}^2 \times \mathbf{C}^*$. In fact, in the holomorphic picture, $q_1=1(q_2=1)$ represents a vanishing volume limit of the exceptional curve $C_1(C_2)$. After a hyperkähler rotation, these vanishing volumes are identified as the corresponding vanishing of the Lagrangian cycles L_1, L_2.

We remark that the above Proposition 3.1, and also the interpretation made above generalize to arbitrary μ in a straightforward way. For example, the relation (3.5) should be read,

$$\sum_{k=0}^{k=\mu+1} a_{k+1} W^k \quad = (1+W)(1+q_1W)(1+q_1q_2W)\cdots(1+q_1\cdots q_\mu W) \ ,$$

and the discriminant is essentially given by the difference products the roots $1, -1/q_1$, $\cdots$, $-1/(q_1\cdots q_\mu)$ (, see Proposition 4.1 and 4.4).

In the rest of this subsection, we look at the mirror map (3.6), and summarize its monodromy property. Let us first note that the moduli space $\mathcal{M}_{Sec(\Sigma)}$ is a two dimensional singular toric variety, and may be desingularized to $\hat{\mathcal{M}}_{Sec(\Sigma)} = Bl_4(\mathbf{P}^1 \times \mathbf{P}^1)$ after blowing up four points (, see the dashed lines in Fig.1). Then the discriminant $dis(x,y)=0$ describes a nodal elliptic curve written in Fig.2. Note that the mirror map form (q_1,q_2) to (x,y) is six to one at generic points because (3.6) is invariant under the reflections;

$$r_1: q_1 \to 1/q_1 \ , \ q_2 \to q_1q_2 \qquad , \qquad r_2: q_2 \to q_1q_2 \ , \ q_1 \to 1/q_2 \ . \tag{3.7}$$

which satisfy $r_1^2 = r_2^2 = (r_1r_2)^3 = 1$ and thus generate the symmetric group of order 3. By making an analytic continuation of the hypergeometric series, we can derive the above invariance group actions (3.7) as the monodromy actions for the loops r_1 and r_2 depicted in Fig.2. Together with the monodromy matrices R_x, R_y about the large complex structure limit, we summarize the monodromy generators with respect to a basis ${}^t(1, w_1(x,y), w_2(x,y)) = {}^t(1, \frac{1}{2\pi i}\log q_1, \frac{1}{2\pi i}\log q_2)$;

$$r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \ , \ r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \ ; \ R_x = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \ .$$

In general, we may state the monodromy property as follows;

Proposition 3.2. *The mirror map (3.6) uniformizes the solutions of the GKZ system (3.3) with the symmetric group $\mathcal{S}_\mu$ of order μ, up to the shifts $w_i(x_1,\cdots,x_\mu) \mapsto w_i(x_1,\cdots,x_\mu)+1$ $(i=1,\cdots,\mu)$. In other words, the mirror coordinate $q_1,\cdots,q_\mu$ uniformizes the hypergeometric series with the symmetric group $\mathcal{S}_\mu$ of order μ* [1].

The above result seems to be interesting from the viewpoint of the uniformization of hypergeometric series[Yo], since our GKZ system provides an infinite number of examples for which multi-valued hypergeometric series are uniformizable. However the result itself is not surprising, as we will show in the next section that our GKZ system has a close relation to K. Saito's differential equations for which the monodromy property is well-studied.

[1]The author was pointed out by Prof. K.Saito that the monodromy group of this GKZ system is the affine Weyl group of the root system A_μ. This is the mirror dual to the affine Weyl group which appears in the recent study of Auteq $D(\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}})$ in [IU].

4. K.Saito's differential equations

In the deformation theory of a singularity, we have a notion of so-called *primitive form* which satisfies a set of differential equations, K. Saito's differential equations[Sa]. Primitive form is also referred to as Gel'fand-Leray form[AGV]. Here we briefly introduce the primitive form and K. Saito's differential equations, and then connect them to the 'period integral' (3.1) and the GKZ system (3.3).

(4-1) K.Saito's system for a primitive form $\mathcal{U}_0(a)$. Let us first note that, in the deformation of the singularity $U^2+V^2+W^{\mu+1}$, the polynomial equation $f_\Sigma(a,W)+U^2+V^2=0$ will be considered in $\mathbf{C}^3$ and the deformation parameters are set to $a_{\mu+2}=1, a_{\mu+1}=0$ by a coordinate change of W. Namely we take the defining equation of the form $f_\Sigma(a,W)+U^2+V^2=a_1+f_1(a,U,V,W)$ with

$$f_1(a,U,V,W)=a_2W+\cdots+a_\mu W^{\mu-1}+W^{\mu+1}+U^2+V^2 \ ,$$

and regard the parameters $a_1,a_2,\cdots,a_\mu$ as giving a deformation of the singularity $W^{\mu+1}+U^2+V^2=0\subset\mathbf{C}^2$ at the origin. Since the parameter a_1 plays a distinguished role from the others, we set the local parameters $(a_2,\ldots,a_\mu)$ as a coordinate of $T:=\mathbf{C}^{\mu-1}$. The full parameters $(a_1,a_2,\cdots,a_\mu)$ will be regarded as a coordinate of $S:=\mathbf{C}\times T$. We consider the product space $\mathfrak{X}:=\mathbf{C}^3\times T$ with coordinate $(U,V,W,a_2,\cdots,a_\mu)$. Then we have a natural map $\varphi:\mathfrak{X}\rightarrow S$ by

$$(U,V,W,a_2,\cdots,a_\mu)\mapsto(-f_1(a,U,V,W),a_2,\cdots,a_\mu) \ .$$

This map plays important roles in describing the deformation of the singularity. Consider the sheaf $\Omega^p_{\mathfrak{X}/T}$ of germs of relative holomorphic p forms for the natural projection $\pi:\mathfrak{X}=\mathbf{C}^3\times T\rightarrow T$. We may consider the following sheaves on S,

$$\mathcal{H}^{(0)}=\varphi_*\Omega^3_{\mathfrak{X}/T}/df_1\wedge d(\varphi_*\Omega^1_{\mathfrak{X}/T}) \ , \ \ \mathcal{H}^{(-1)}=\varphi_*\Omega^2_{\mathfrak{X}/T}/(df_1\wedge\varphi_*\Omega^1_{\mathfrak{X}/T}+d(\varphi_*\Omega^1_{\mathfrak{X}/T})) \ .$$

A primitive form ζ is an element in $H^0(S,\mathcal{H}^{(0)})$ satisfying certain conditions (, see [Sa] for details). Instead of ζ, hereafter, we consider its image $\mathcal{U}_0$ in $H^0(S,\mathcal{H}^{(-1)})$ under an isomorphisms $\mathcal{H}^{(0)}\cong\mathcal{H}^{(-1)}$ (see [Sa]), which we may write explicitly as

$$\mathcal{U}_0(a)=Res_{\{a_1+f_1=0\}}\left(\frac{dU\wedge dV\wedge dW}{a_1+f_1(a,U,V,W)}\right) \ .$$

In this from, the primitive form is also called as the Gel'fand-Leray form in the study of oscillating integrals (, see e.g. [AGV]). We note that the similarity of $\mathcal{U}_0(a)$ to our 'period integral' (3.1), although, in (3.1), we do not set $a_{\mu+2}=1, a_{\mu+1}=0$ but consider torus actions $(\mathbf{C}^*)^2$ instead.

K.Saito's system is defined as a set of differential equations satisfied by the primitive form $\mathcal{U}_0(a)=\mathcal{U}_0(a_1,\cdots,a_\mu)$;

$$\begin{aligned} P_{ij}\mathcal{U}_0(a) \quad &= \left\{\frac{\partial^2}{\partial a_i\partial a_j}-\nabla_{\frac{\partial}{\partial a_i}}\frac{\partial}{\partial a_j}-(\frac{\partial}{\partial a_i}*\frac{\partial}{\partial a_j})\frac{\partial}{\partial a_1}\right\}\mathcal{U}_0(a)=0 \ , \\ Q(\tfrac{\partial}{\partial a_i})\mathcal{U}_0(a) \quad &= \left\{w(\frac{\partial}{\partial a_i})\frac{\partial}{\partial a_1}-N(\frac{\partial}{\partial a_i})+\frac{3}{2}\frac{\partial}{\partial a_i}\right\}\mathcal{U}_0(a)=0 \ , \end{aligned}$$

see [Sa] for detailed definitions. This system is defined for a general setting in the deformation theory of singularities, and also known to define a holonomic system. A proof of the following proposition may be found in Appendix by Ambai in [Oda1], for example. Although, we restrict our attention to A_μ case, similar results are known also for other (D or E) types of singularities.

Proposition 4.1. *Let $\beta_0(a), \cdots, \beta_\mu(a)$ be the roots of $\psi_0(W) := a_1 + a_2 W + \cdots + a_\mu W^{\mu-1} + W^{\mu+1}$ which satisfy $\beta_0(a) + \cdots + \beta_\mu(a) = 0$. Then the space of solutions of K.Saito's system is generated by*

$$1 \ , \ \beta_0(a) - \beta_1(a) \ \ , \ \ \cdots \ , \ \beta_{\mu-1}(a) - \beta_\mu(a) \ \ . \tag{4.1}$$

The system has a regular singularity at the discriminant locus

$$dis\psi_0(a) = \Pi_{1 \le i,j \le \mu}(\beta_i(a) - \beta_j(a))^2 = 0 \ \ ,$$

and the monodromy group about the discriminant coincides with the symmetric group S_μ of order μ acting as permutations among the roots $\beta_i(a)$.

As we have noted in (3-1), μ vanishing cycles appears in the deformation of the singularity. Now it is easy to find that the above solutions $\alpha_i := \beta_i(a) - \beta_{i-1}(a)$ represent the integrals $\int_C \mathcal{U}_0(a)$ over the corresponding vanishing cycles (, see Appendix A). Also we note that there is a residue pairing $I(\alpha_i, \alpha_j)$ among the solutions which reproduces the intersection pairing $\# L_i \cap L_j$ among the vanishing cycles [Sa].

(4-2) GKZ system for a primitive form $\mathcal{U}(a)$. As defined above, the primitive form $\mathcal{U}_0(a)$ is parametrized by $(a_1, \cdots, a_\mu) \in S$ by setting $a_{\mu+1} = 0, a_{\mu+2} = 1$ in the defining equation $f_\Sigma(a, W) + U^2 + V^2$. Instead of this, one may consider a natural torus actions $(\mathbf{C}^*)^2$ on $(a_1, \cdots, a_{\mu+2})$ by $a_i \mapsto \lambda a_i$ and $a_i \mapsto \lambda^{i-1} a_i$ ($\lambda \in \mathbf{C}^*$). With this slight change of the parameters ('gauge'), we may connect K.Saito's system to a GKZ system. Let us define a period integral of the primitive form

$$\int_C \mathcal{U}(a) = \int_C Res_{f_\Sigma(W)+U^2+V^2=0} \left(\frac{dW \wedge dU \wedge dV}{f_\Sigma(a,W) + U^2 + V^2} \right) \ \ , \tag{4.2}$$

where C is a two cycle on $f_\Sigma(a, W) + U^2 + V^2 = 0 \subset \mathbf{C}^3$ and $f_\Sigma(a, W) = a_1 + a_2 W + \cdots + a_{\mu+2} W^{\mu+1}$. The period integral above is similar to (3.1), and thus satisfies a similar GKZ system to (3.3). The only difference appears in the scaling properties expressed by the linear operators $\mathcal{Z}_i$.

Proposition 4.2. *1) The period integral (4.2) satisfies*

$$\Box_l \int_C \mathcal{U}(a) = 0 \ (l \in \mathcal{L}) \ \ , \ \ \mathcal{Z}'_i \int_C \mathcal{U}(a) = 0 \ (i = 1, 2) \ , \tag{4.3}$$

where the operators $\Box_l$ and the lattice $\mathcal{L}$ are the same as in (3.3), and $\mathcal{Z}'_i$ $(i = 1, 2)$ are given by

$$\begin{pmatrix} \mathcal{Z}'_1 \\ \mathcal{Z}'_2 \end{pmatrix} = \begin{pmatrix} \theta_1 + \theta_2 + \cdots + \theta_{\mu+2} \\ \theta_2 + 2\theta_3 \cdots + (\mu+1)\theta_{\mu+2} + 1 \end{pmatrix}$$

2) The system (4.3) above is reducible of rank $\mu+1$ with its irreducible part of rank μ. The μ independent solutions of the irreducible part are given by

$$\beta_0(a) - \beta_1(a), \cdots, \beta_{\mu-1}(a) - \beta_\mu(a) \ \ ,$$

where $\beta_i(a)$ are roots of $\psi(W) = a_1 + a_2 W + \cdots + a_{\mu+2} W^{\mu+1} = 0$.

Derivation of 1) above is straightforward, but it should be noted that K.Saito's system is replaced by a different but simpler GKZ system. Also, it is known that the numbers of independent solutions of a GKZ system is given by the volume of a

relevant polytope[GKZ1], which in our case is $Vol(\Sigma) = \mu+1$. One may verify that the system is reducible observing the factorization of a differential operator in our GKZ system when expressing $\Box_l$ operators in the affine coordinates of $\mathcal{M}_{Sec(\Sigma)}$. After the factorization, the reduced GKZ system has μ independent solutions as claimed. (It has been observed in [HKTY1] that the GKZ systems in mirror symmetry are often reducible in this way.) One can derive the μ independent solutions in 2) by evaluating the period integral (4.2) over the vanishing cycles (, see Appendix A). Or more directly, one may verify that the roots β_k's are solutions of (4.3) with one relation $\beta_0 + \cdots + \beta_\mu = -\frac{a_{\mu+1}}{a_{\mu+2}}$ from the following property:

Proposition 4.3. *The root $\beta(a)\,(=\beta_0(a), \cdots, \beta_\mu(a))$ of $\psi(W) = 0$ satisfies*

$$\frac{\partial \beta}{\partial a_i} = -\frac{\beta^{i-1}}{\psi'(\beta)} \;\;,\quad \frac{\partial^2 \beta}{\partial a_i \partial a_j} = \frac{1}{\psi'(\beta)} \frac{d}{dx}\left(\frac{x^{i+j-2}}{\psi'(x)}\right)\Big|_{x=\beta} . \tag{4.4}$$

Proof. The first relation follows from the differentiation with respect to a_i of $\psi(\beta(a)) = a_1 + a_2\beta(a) + \cdot + a_{\mu+2}\beta(a)^{\mu+1} = 0$. For the second relation, we note the following;

$$\frac{\partial \psi'(\beta)}{\partial a_j} = (j-1)\beta^{j-2} + \psi''(\beta)\frac{\partial \beta}{\partial a_j} = (j-1)\beta^{j-2} - \frac{\psi''(\beta)}{\psi'(\beta)}\beta^{j-1} .$$

Then the second relation follows by differentiating the first equation with respect to a_i. □

Now a proof of Proposition 4.2 is straightforward by the relations above. Here, since the arguments are similar, we present the solutions of the GKZ system (3.3) for the local mirror symmetry, which is the same as the GKZ system (4.3) for the primitive form except the scalings operators.

Proposition 4.4. *1) The independent solutions of the GKZ system (3.3) are given by*

$$1 \;\;,\;\; \log\beta_0(a) - \log\beta_1(a) \;\;,\;\; \cdots \;\;,\;\; \log\beta_{\mu-1}(a) - \log\beta_\mu(a) \;\; .$$

2) These solutions, up to monodromy transformations, are related to the expansion $w(x; \frac{J}{2\pi i}) = 1 + \sum_{k=1}^{\mu} w_k(a) J_k$ near the large complex structure by

$$2\pi i\, w_k(x) = \log\beta_{k-1}(a) - \log\beta_k(a) \;\; (k = 1, \cdots, \mu).$$

Proof. 1) By the relations (4.4), it it straightforward to evaluate $\mathcal{Z}_1 \log\beta$ and $\mathcal{Z}_2 \log\beta$ as follows;

$$\mathcal{Z}_1 \log\beta = \frac{1}{\beta}\sum_{i=1}^{\mu+2} \theta_{a_i}\beta = -\frac{1}{\beta}\sum_{i=1}^{\mu+2}\frac{1}{\psi'(\beta)} a_i \beta^{i-1} = -\frac{1}{\beta\psi'(\beta)}\psi(\beta) = 0 \;\; ,$$

$$\mathcal{Z}_2 \log\beta = \frac{1}{\beta}\sum_{i=1}^{\mu+2}(i-1)\theta_{a_i}\beta = -\frac{1}{\psi'(\beta)}\sum_{i=1}^{\mu+2}(i-1)a_i\beta^{i-2} = -1 \; .$$

Therefore the differences $\log\beta_k - \log\beta_{k-1}$ $(k = 1, .., \mu)$ are annihilated by the linear operators $\mathcal{Z}_1, \mathcal{Z}_2$. As for the operators $\Box_l$ $(l \in \mathcal{L})$, we first note

$$\frac{\partial^2}{\partial a_i \partial a_j}\log\beta = \frac{\partial^2}{\partial a_k \partial a_l}\log\beta \quad \text{if} \;\; i+j = k+l \; , \tag{4.5}$$

which follows from
$$\frac{\partial^2}{\partial a_i \partial a_j} \log \beta = -\frac{1}{\beta^2} \frac{\beta^{i+j-2}}{(\psi'(\beta))^2} + \frac{1}{\beta} \frac{1}{\psi'(\beta)} \frac{d}{dx} \left(\frac{x^{i+j-2}}{\psi'(x)} \right) \Big|_{x=\beta} .$$
Since the entries of $l = (l_1, l_2, ..., l_{\mu+2})$ satisfies $\sum_i l_i = 0$ and $\sum_i (i-1) l_i = 0$, we have $\sum_i i\, l_i^+ = \sum_i i\, l_i^-$ for $l_i^{\pm} := \frac{1}{2}(l_i \pm |l_i|)$. Now, by repeated use of (4.5), one may deduce
$$\prod_i \left(\frac{\partial}{\partial a_i} \right)^{l_i^+} \log \beta = \prod_i \left(\frac{\partial}{\partial a_i} \right)^{l_i^-} \log \beta \ ,$$
which means $\Box_l \log \beta = 0$. Since the linear independence of $\log \beta_k - \log \beta_{k-1}$ ($k = 1, .., \mu$) is clear, we obtain the claim including the trivial solution 1.

2) By definition, we have $2\pi i\, w_k(x) \sim \log x_k$, where $x_k = \frac{a_k a_{k+2}}{a_{k+1}^2}$. Since, by extending the classical arguments due to Frobenius, it is straightforward to show that $w_k(x)$ are solutions of the GKZ hypergeometric system (3.3), we only need to connect these solutions to those in terms of roots β_k. For this, let us assume that the roots are given by $\beta_k = -\frac{1}{\lambda_k}$ and also the following relation,
$$\psi(W) = a_1 + a_2 W + \cdots + a_{\mu+2} W^{\mu+1} = \prod_k (\lambda_k W + 1) \ .$$
Assume also that $|\lambda_\mu| << |\lambda_{\mu-1}| << \cdots << |\lambda_0|$. Then from the relations between roots and the coefficients, it is easy to see
$$\log a_1 = 0 \ , \ \log a_2 = \log \lambda_0 + A_2(\lambda) \ , \ \log a_3 = \log(\lambda_0 \lambda_1) + A_3(\lambda) \ , \ \cdots \ ,$$
$$\log a_{\mu+2} = \log(\lambda_0 \lambda_1 \cdots \lambda_\mu) + A_{\mu+2}(\lambda) \ ,$$
where $A_k(\lambda) \in \mathbf{C}\{\frac{\lambda_1}{\lambda_0}, \frac{\lambda_2}{\lambda_1}, \cdots, \frac{\lambda_\mu}{\lambda_{\mu-1}}\}$. Now we have
$$\log x_k = \log \frac{a_k a_{k+2}}{a_{k+1}^2} = \log \lambda_k - \log \lambda_{k-1} + B_k(\lambda) \ ,$$
with $B_k(\lambda) := A_k(\lambda) + A_{k+2}(\lambda) - 2A_{k+1}(\lambda)$, which entails the claimed relation since $2\pi i w_k(x) \sim \log x_k$. (Note that if we invert the series relations $x_k = \frac{\lambda_k}{\lambda_{k-1}} e^{B_k(\lambda)}$ in $C\{\frac{\lambda_1}{\lambda_0}, \frac{\lambda_2}{\lambda_1}, \cdots, \frac{\lambda_\mu}{\lambda_{\mu-1}}\}$, then we obtain $\frac{\lambda_k}{\lambda_{k-1}}$'s in terms of x_k's, and thus recover the series expansion of $2\pi i w_k(x)$ by $2\pi i w_k(x) = \log \lambda_k - \log \lambda_{k-1} = \log x_k - B_k(\lambda)$.) $\Box$

The above form of the solutions may be connected to the piecewise linear functions on the fan Σ [Ba3]. Since we have established a relation of the GKZ system (3.3) of $\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ to K.Saito's system of the primitive form, for which integral monodromy property is well studied (see Proposition 4.1), it is clear that we have the uniformization property of the mirror map stated in Proposition 3.2.

5. **Central charge formula and G-Hilb**

In the last two sections, we have looked at the local mirror symmetry of $\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ paying our attentions to the monodromy property of the associated GKZ system. Here we come back to our claim for the central charge formula (2.4) in this case.

Let us first recall that the non-compact Calabi-Yau manifold $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ is given as the Hilbert scheme of points on $\mathbf{C}^2$, G-Hilb $\mathbf{C}^2$. Here G-Hilb $\mathbf{C}^n$ is defined

for a finite subgroup $G \subset SL(n, \mathbf{C})$ and consists of zero dimensional subschemes Z in $\mathbf{C}^n$ of length equal $|G|$ such that G acts on Z and $H^0(\mathcal{O}_Z)$ as the regular representation of G. The following results are due to Gonzalez-Springberg and Verdier[GV] for $n = 2$ and their generalizations to $n = 3$ have been done in [Na][IN][BKR]. Here, we summarize the relevant results for our cases where $G \subset SL(n, \mathbf{C})$ is a finite abelian subgroup:

(i) The K-group $K(X)$ of algebraic vector bundles on X are generated by the so-called tautological bundles $\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_d$ $(d = |G| - 1)$, where the subscripts refer to the one-dimensional representations of G with 0 for the trivial representation.

(ii) Let $K^c(X)$ be the K-group of the complexes of algebraic vector bundles which are exact off $\pi^{-1}(0)$ where $\pi : X \to \mathbf{C}^n/G$. Then there exists a complete pairing $K^c(X) \times K(X) \to \mathbf{Z}$, and the dual basis $S_0, S_1, \cdots, S_d$ of $K^c(X)$ satisfying

$$\langle ch(S_i), ch(\mathcal{F}_j) \rangle := \int_X ch(S_i) ch(\mathcal{F}_j) Todd_X = \delta_{ij} \quad . \tag{5.1}$$

(iii) The dual basis $S_k (k = 0, \cdots, d)$ defines a symmetric form for $n = 2$ (, a symplectic form for $n = 3$,) on $K^c(X)$ by

$$\chi(S_i, S_j) := \int_X ch(S_i^\vee) ch(S_j) Todd_X \ \ (0 \le i, j \le d). \tag{5.2}$$

In two dimensional case with $G = \mathbf{Z}_{\mu+1}$, S_k's for $k \neq 0$ have a simple forms. They are given by $S_k = \mathcal{O}_{C_k}(-1)$ with $C_k \cong \mathbf{P}^1$ being the exceptional curves. Including S_0, they satisfy $\chi(S_i, S_j) = -\hat{\mathcal{C}}_{ij}$ $\ (0 \le i, j \le d)$ where $\hat{\mathcal{C}}_{ij}$ is the extended Cartan matrix of the root system of type A_μ.

Now with these properties about the geometry of $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$, let us recall our Conjecture 2.2;

$$w(x_1, \cdots, x_r; \frac{J_1}{2\pi i}, \cdots, \frac{J_r}{2\pi i}) = \sum_{i,j} \int_{mir(E_i)} \Omega(Y_x) \ \chi^{ij} \ ch(E_j^\vee) \ .$$

One should note that this is a conjecture for X a compact Calabi-Yau manifold. It is rather clear how to modify this relation for our non-compact X if we notice that the expression $\sum_j \chi^{ij} ch(E_j^\vee)$ satisfies $\int_X ch(E_k) \sum_j \chi^{ij} ch(E_j^\vee) Todd_X = \delta^i{}_k$, i.e., provides a dual base to $ch(E_k)$. From this, we claim that the central charge formula for $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ is given by

$$w(x_1, \cdots, x_\mu; \frac{J_1}{2\pi i}, \cdots, \frac{J_\mu}{2\pi i}) = \sum_k \int_{mir(S_k)} \Omega(Y_x) \ ch(\mathcal{F}_k) \ , \tag{5.3}$$

where the holomorphic two form $\Omega(Y_x)$ should be understood as a holomorphic two-form associated to the integrand of (3.1), i.e.,

$$\Omega(Y_x) = \frac{i}{2\pi^2} Res \left(\frac{1}{f(W) + U^2 + V^2} dU dV \frac{dW}{W} \right) \quad .$$

We will state the corresponding claim in Conjecture 6.3 for three dimensional cases.

Now, since the tautological bundles satisfy

$$\int_{C_j} c_1(\mathcal{F}_k) = \delta_{jk} \ \ (1 \le j, k \le \mu),$$

for the exceptional curves $C_j\,(1 \le j \le \mu)$, we have $c_1(\mathcal{F}_k) = J_k$ $(k \ge 1)$ for our generators J_k of $H^2(X, \mathbf{Z})$. Therefore we have

$$w(x; \frac{J}{2\pi i}) = 1 + \sum_k w_k(x) J_k = (1 - \sum_{k=1}^{\mu} w_k(x)) ch(\mathcal{F}_0) + \sum_{k=1}^{\mu} w_k(x) ch(\mathcal{F}_k) \ ,$$

where we use $ch(\mathcal{F}_k) = 1 + c_1(\mathcal{F}_k) = 1 + J_k$. From this, one may read the central charge of the sheaf $S_k = \mathcal{O}_{C_k}(-1)\,(k \ne 0)$ as

$$Z_t(S_k) = \int_{mir(S_k)} \Omega(Y_x) = w_k(x) = \frac{1}{2\pi i}\big(\log \beta_{k-1}(a) - \log \beta_k(a)\big) \ . \tag{5.4}$$

In Appendix A, we obtain the above central charge as a period integral over a vanishing cycle. Therefore, we have obtained a refinement of the mirror symmetry sketched in (3-1) that the mirror image $mir(S_k)$ of the sheaf $S_k = \mathcal{O}_{C_k}(-1)$ is a vanishing cycle whose period is given by $Z_t(S_k)$ above.

Similar relation holds also for S_0 with its central charge $1 - \sum_k w_k(x)$. However, since S_0 is not a sheaf but a complex of sheaves (see [IN]), we replace it with the skyscraper sheaf $\mathcal{O}_p$ supported at a point p. Then the sheaves $\mathcal{O}_p, \mathcal{O}_{C_k}(-1)$ $(k = 1, \cdots, \mu)$ form another basis of $K^c(X)$ with relations

$$\mathcal{O}_p = S_0 + S_1 + \cdots + S_\mu \ , \ \ \mathcal{O}_{C_k}(-1) = S_k \ (k = 1, \cdots, \mu) \ ,$$

where we use $\langle \mathcal{O}_p, \mathcal{F}_j \rangle = 1$ $(j = 0, \cdots, \mu)$ to derive the first equality. In (6-2), this basis of $K^c(X)$ will be generalized as a *symplectic D-brane basis* to the three-dimensional cases. Now, for the sheaf $\mathcal{O}_p$, we have

$$Z_t(\mathcal{O}_p) = \int_{mir(\mathcal{O}_p)} \Omega(Y_x) \equiv 1 \ .$$

This is identified with the period integral over a cycle T_0 whose topology is $S^1 \times S^1$, see Appendix A.

Finally, we note that the sheaves S_k $(k \ne 0)$ are spherical and thus define self-equivalences of $D^b(Coh(X))$, the Seidel-Thomas twists summarized in section 2. In [ST, Proposition 3.19], it is shown that these spherical objects, which form the so-called (A_μ)-configuration, generate a weak braid group action on $D^b(Coh(X))$. This braid group action should be mirror to the corresponding Dehn twists (Picard-Lefschetz transformations) in the symplectic side. Our formula (5.4) explicitly shows this mirror correspondence as the linear transformations on the central charges.

6. Local mirror symmetry II — $X = \widehat{\mathbf{C}^3/G}$

In the two dimensional cases, the local mirror symmetry is rather clear from the hyperkähler rotation as summarized in (3-1), and our results in the previous sections are consistent with those arguments from the hyperkähler rotation. In this section, we will argue that our central charge formula for a crepant resolution of the three dimensional singularity $\mathbf{C}^3/G$ with a finite abelian group $G \subset SL(3, \mathbf{C})$. The crepant resolution we use is the so-called G-Hilb $\mathbf{C}^3$ whose definition has been given in the previous section. Examples of GKZ hypergeometric system in this situation have been studied also in [dOFS].

(6-1) Period integrals and GKZ systems. As above, let us consider a non-compact Calabi-Yau manifold $X = G$-Hilb $\mathbf{C}^3$ with a finite, abelian group $G \subset SL(3, \mathbf{C})$. For G being abelian, X is given as a toric resolution of a singularity $\mathbf{C}^3/G$ as follows: Let us first denote by M_G the lattice corresponding to the invariants $\mathbf{C}[x^{\pm}, y^{\pm}, z^{\pm}]^G$, and its dual lattice by N_G. Then the invariant monomials in $\mathbf{C}[x, y, z]^G$ defines a cone in $M_G \otimes \mathbf{R}$, with its dual cone Σ_G^0 in $N_G \otimes \mathbf{R}$. The toric (Calabi-Yau) resolution which corresponds to G-Hilb $\mathbf{C}^3$ is given by a fan obtained by subdividing the cone Σ_G^0, which we will denote by Σ_G hereafter. (See [Re] for detailed construction of the fan Σ_G.) We will denote by $\nu_1, \nu_2, \cdots, \nu_{r+3}$ the integral generators of the one-dimensional cones in the fan Σ_G. Note that, since the resolution is crepant, these generators satisfy

$$(m, \nu_k) = 1 \ \ (k = 1, \cdots, r+3) ,$$

with some $m \in M_G$, where $(\ ,\ ) : M_G \times N_G \to \mathbf{Z}$ is the dual pairing.

To describe the mirror configuration of X, let us fix an isomorphism $\varphi : N_G \simeq \mathbf{Z}^3$ satisfying $\varphi(\nu_k) = (1, \bar{\nu}_k)$. Consider the defining equation

$$F(a; U, V, W_1, W_2) = U^2 + V^2 + \sum_k a_k W^{\bar{\nu}_k} \ \ , \tag{6.1}$$

then the mirror configuration of X claimed in [CKYZ][HIV] is the hypersurface $Y = (F(a; U, V, W_1, W_2) = 0) \subset \mathbf{C}^2 \times (\mathbf{C}^*)^2$. Note that Y does not depend on the choice of the isomorphism $\varphi : N_G \tilde{\to} \mathbf{Z}^3$ with the described property, since W_1, W_2 are considered in $\mathbf{C}^*$ and negative powers of them are allowed. One important criterion for the mirror configuration Y is the fact that we can extract the right Gromov-Witten invariants for X from the "period integrals" of Y (cf. [CKYZ]).

Definition 6.1. For the mirror configuration Y given above, we define the period integral of a cycle $L \in H_3(Y, \mathbf{Z})$ by

$$\Pi_L(a) = \frac{1}{4\pi^3} \int_L Res_{F=0} \left(\frac{1}{U^2 + V^2 + \sum_k a_k W^{\bar{\nu}_k}} dU dV \frac{dW_1}{W_1} \frac{dW_2}{W_2} \right) , \tag{6.2}$$

where the residue is taken around $F(a, U, V, W) = 0$ in $\mathbf{C}^2 \times (\mathbf{C}^*)^2$.

In (6-3), we will describe how one can determine the Gromov-Witten invariants from our cohomology-valued hypergeometric series. Here, as in the two dimensional cases, we see that the following GKZ system is satisfied by the period integral:

Proposition 6.2. *Let $\mathcal{L}$ be the lattice of relations defined by*

$$\mathcal{L} = \{(l_1, \cdots, l_{r+3}) \in \mathbf{Z}^{r+3} \mid \sum_k l_k \nu_k = \vec{0}\ \}.$$

Then the period integral (6.2) satisfies the following set of differential equations:

$$\Box_l \Pi_L(a) = 0 \ (l \in \mathcal{L}) \ \ , \ \ \mathcal{Z}_i \Pi_L(a) = 0 \ (i = 1, 2, 3) \ , \tag{6.3}$$

where

$$\Box_l = \left(\frac{\partial}{\partial a}\right)^{l^+} - \left(\frac{\partial}{\partial a}\right)^{l^-} \quad , \quad \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \mathcal{Z}_3 \end{pmatrix} = \begin{pmatrix} \theta_1 + \theta_2 + \cdots + \theta_{r+3} \\ \sum_k \bar{\nu}_{k,2}\theta_k \\ \sum_k \bar{\nu}_{k,3}\theta_k \end{pmatrix}$$

with $l = l^+ - l^-$ ($l_i^\pm := \frac{1}{2}(l_i \pm |l_i|)$) and $\theta_k = a_k \frac{\partial}{\partial a_k}$. $\bar{\nu}_{k,i}$ represents the i-th component of the vector $\varphi(\nu_k) = (1, \bar{\nu}_k)$ with the isomorphism $\varphi : N_G \xrightarrow{\sim} \mathbf{Z}^3$ fixed above.

Remark. As we have shown in section 4, by a slight modification of the period integral, we can connect (6.2) to the K. Saito's primitive form of a three dimensional singularity. The primitive form satisfies the same GKZ hypergeometric system but with a different scaling property, i.e., $\mathcal{Z}_i (i = 2, 3)$ being replaced by $\mathcal{Z}'_i := \mathcal{Z}_i + 1 (i = 2, 3)$. One may observe for several examples that, when the singularity of $\mathbf{C}^3/G$ is isolated, the resulting GKZ system for the primitive form is reducible and its irreducible part shows a slightly different degeneration at 'infinity', i.e. at the large complex structure limit (, see Appendix B). $\Box$

(6-2) Central charge formula and symplectic forms. The cohomology-valued hypergeometric series for the GKZ system (6.3) are defined by constructing the secondary fan $Sec(\Sigma_G)$. As described for an example in (3-3) and Fig.1, the secondary fan is constructed combinatorially from the secondary polytope in (a translate of) $\mathcal{L}_{\mathbf{R}}$. We refer [HLY],[Hos],[Sti] and references therein for details of the construction in our context of mirror symmetry.

Let us denote by $l^{(1)}, \cdots, l^{(r)}$ the integral generators of $\mathcal{L}$ which represents the large complex structure limit. We denote the affine coordinate by $x_k = a^{l^{(k)}}$ ($k = 1, \cdots, r$). Then, we have cohomology-valued hypergeometric series

$$w(x_1, \cdot\cdot, x_r; \frac{J_1}{2\pi i}, \cdot\cdot, \frac{J_r}{2\pi i}) = \sum_{m \in \mathbf{Z}^s_{\geq 0}} c(m + \rho) x^{m+\rho}|_{\rho = \frac{J}{2\pi i}}, \tag{6.4}$$

where $c(m) := c(m_1, \cdots, m_r)$ is defined by

$$c(m_1, \cdots, m_r) := \frac{1}{\prod_{i=1,\ldots,r+3} \Gamma(1 + \sum_k m_k l_i^{(k)})} \ ,$$

and $J_1, \cdots, J_r \in H^2(X, \mathbf{Z})$ are dual (semi-ample) generators to $l^{(1)}, \cdots, l^{(r)}$ (, see [Sect.1, Hos] for a quick review of the construction). It is found in [HKTY2] for X compact Calabi-Yau hypersurfaces (see also [Sti]), and used in [CKYZ] for non-compact cases that the above cohomology-valued hypergeometric series generate the solutions of the GKZ system (6.3) when expanded in $H^{even}(X, \mathbf{Q})$. If we recall the integral structure of $K(X)$ and also the natural symplectic structure in $K^c(X)$ which are summarized in section 4, we come to the following conjecture:

Conjecture 6.3. *1) Consider the expansion of the cohomology-valued hypergeometric series (6.4) with respect to the basis* $ch(\mathcal{F}_0), \cdots, ch(\mathcal{F}_d)$ *of* $H^{even}(X, \mathbf{Q})$*;* $w(x; \frac{J}{2\pi i}) = \sum_k f_k(x) ch(\mathcal{F}_k)$*. Then the coefficient hypergeometric series* $f_k(x)$ *may be identified with the period integrals over the cycles* $mir(S_k)$*, i.e., we have*

$$w(x_1, \cdots, x_r; \frac{J_1}{2\pi i}, \cdots, \frac{J_r}{2\pi i}) = \sum_k \int_{mir(S_k)} \Omega(Y_x) \; ch(\mathcal{F}_k) \;, \tag{6.5}$$

where $mir : K^c(X) \tilde{\rightarrow} H_3(Y, \mathbf{Z})$ *is the (homological) mirror map and* $\Omega(Y_x) = \frac{1}{4\pi^3} Res_{F=0}(\frac{1}{F(a,U,V,W)} dU dV \frac{dW_1}{W_1} \frac{dW_2}{W_2})$*.*
2) The monodromy of the coefficient hypergeometric series $f_k(x) (= \int_{mir(S_k)} \Omega(Y_x))$ *is integral and symplectic with respect to the symplectic form;*

$$\chi(S_i, S_j) = \int_X ch(S_i^\vee) ch(S_j) Todd_X \quad (0 \le i, j \le d = |G| - 1).$$

3) Using the cohomology-valued hypergeometric series, the central charge of $F \in K^c(X)$ *is given by* $Z_t(F) = \int_X ch(F) w(x; \frac{J}{2\pi i}) Todd_X$*.*

In section 4 and section 5, we have verified explicitly the corresponding conjecture for $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ by connecting our period integral to that of the primitive form in singularity theory. For three dimensions, however, little is known about the explicit form of the period integral of the primitive form. In the following sections, we will test our conjecture for the cases, $G = \mathbf{Z}_3, \mathbf{Z}_5$. In particular, for the former case, we will evaluate the period integrals of the primitive form, and connect them to the periods in mirror symmetry (, see Appendix A). In this case, we will be able to see the mirror correspondence $mir : K^c(X) \tilde{\rightarrow} H_3(Y, \mathbf{Z})$ from our Conjecture 6.3, which generalizes the correspondence between the spherical objects and the vanishing cycles presented in section 5.

As another aspect of our conjecture, we will define the so-called prepotential which gives predictions of Gromov-Witten invariants on the geometry X. When X is compact, the existence of the prepotential (special Kähler geometry) is ensured by a canonical symplectic basis of $H_3(Y, \mathbf{Z})$ and also by the property called Griffiths transversality [St]. Correspondingly, in our non-compact cases, we have the symplectic structure on $H_3(Y, \mathbf{Z})$ given by $(\chi(S_i, S_j))$ under the homological mirror map, $mir : K^c(X) \tilde{\rightarrow} H_3(Y, \mathbf{Z})$. We will find that our cohomology-valued hypergeometric series combined with the symplectic structure suffices to present a closed formula of the prepotential.

(6-3) Prepotential and a symplectic D-brane basis. In string theory, the so-called D-branes play important roles for duality symmetries such as mirror symmetry. In our non-compact Calabi-Yau manifolds, roughly speaking, the D-branes are elements in $K^c(X)$ or more precisely elements in the category $D^b_c(Coh(X))$ of compact support. In what follows, we will work on $K^c(X)$(or $H^{even}_c(X, \mathbf{Q})$ taking the Chern character). In our case, it is easy to see that there are natural D-branes like $\mathcal{O}_p$, the skyscraper sheaf, $\mathcal{O}_C$, the torsion sheaf supported on a curve C, and $\mathcal{O}_D$, that supported on an exceptional divisor D. Depending on the dimensions of the support, these sheaves are called $D0$-, $D2$-, $D4$-branes, respectively, see [Do] and references therein. We will connect the bases S_i's to these D-branes.

(1) *Symplectic D-brane basis and its dual*: First, let us recall that we have introduced the (semi-ample) generators $J_1, \cdots, J_r$ of $H^2(X, \mathbf{Z})$ when writing down our

cohomology-valued hypergeometric series. Assume that these generators measure the volume of the respective curves $C_1, \cdots, C_r$, i.e.,

$$\int_{C_i} J_k = \delta_{ik} \quad .$$

Assume also that the exceptional divisors $D_1, \cdots, D_s$ provide a basis of $H_4(X, \mathbf{Z})$. Then in such cases, we seek a basis of $K^c(X)$ of the form,

$$\{\mathcal{B}_I\} := \{\; \mathcal{O}_p\,;\, \mathcal{O}_{C_1}(-J_1), \cdots, \mathcal{O}_{C_r}(-J_r)\,;\, \mathcal{O}_{D_1}(\mathcal{L}_1), \cdots, \mathcal{O}_{D_s}(\mathcal{L}_s) \;\} \tag{6.6}$$

where $\mathcal{O}_C(-J) = \mathcal{O}_C \otimes \mathcal{O}_X(-J)$ and $\mathcal{O}_{D_k}(\mathcal{L}_k) := \mathcal{O}_D \otimes \mathcal{L}_k$ with $\mathcal{L}_k = \mathcal{O}_X(-a_{k,1}J_1 - \cdots - a_{k,r}J_r)$. Here we fix the integers $a_{k,1}, ..., a_{k,r}$ requiring

$$\chi(\mathcal{O}_{D_j}(\mathcal{L}_j), \mathcal{O}_{D_k}(\mathcal{L}_k)) = 0 \tag{6.7}$$

for $j, k = 1, \cdots, s$. The integers $a_{k,1}, ..., a_{k,r}$ satisfying the above requirement are not unique in general. However the reason of our requirement (6.7) will become clear soon when we present a closed formula of the prepotential. Hereafter, we will call a basis $\{\mathcal{B}_I\}$ satisfying the requirement as a *symplectic D-brane basis* of $K^c(X)$.

Now let us consider the dual basis of $K(X)$ to the basis $\{\mathcal{B}_I\}$ (6.6) under the paring (5.1). Taking $ch : K(X) \to H^{even}(X, \mathbf{Q})$, we construct (charges) $\{Q_J\} := \{Q_0; Q_2^1, .., Q_2^r; Q_4^1, .., Q_4^s\}$ which satisfies

$$\int_X ch(\mathcal{B}_I)\, Q_J\, Todd_X = \delta_{IJ} \quad .$$

We note that, due to the support property of the D-brane basis (6.6) and the above relation, the base Q_0 starts from the degree zero element 1 to higher degree terms. Similarly Q_2^a starts from J_a to higher degree terms, and Q_4^b is written by a linear combination of the degree four elements of $H^{even}(X, \mathbf{Q})$. The set of charges $\{Q_J\}$ provides a basis of $H^{even}(X, \mathbf{Q})$.

(2) *Expansion of* $w(x; \frac{J}{2\pi i})$ *and the prepotential*: Now we expand our cohomology-valued hypergeometric series using the basis $\{Q_J\}$ of $H^{even}(X, \mathbf{Q})$;

$$w(x; \frac{J}{2\pi i}) = w_0(x) Q_0 + \sum_{a=1}^{r} w_a^{(1)}(x) Q_2^a + \sum_{b=1}^{s} w_b^{(2)}(x) Q_4^b \quad .$$

Then our conjecture says that the coefficient hypergeometric series $\{w_0(x), w_a^{(1)}(x), w_b^{(2)}(x)\}$ represents the central charges of the sheaves $\{\mathcal{B}_I\}$ in (6.6) and whose monodromy matrices are integral and symplectic with respect to the symplectic form

$$(\chi(\mathcal{B}_I, \mathcal{B}_J)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_{ab} \\ 0 & -C_{ab} & 0 \end{pmatrix}, \tag{6.8}$$

where we set $C_{ab} = \chi(\mathcal{O}_{C_a}(-J_a), \mathcal{O}_{D_b}(\mathcal{L}_b))$ and used the fact $\chi(\mathcal{O}_p, \mathcal{B}_I) = 0$ for sheaves $\mathcal{B}_I$ of compact support. The above form should be contrasted to the canonical symplectic form on $H_3(Y, \mathbf{Z})$ for the compact (mirror) Calabi-Yau manifolds, by which we define the special Kähler geometry on the moduli spaces(, see e.g. [St]). In fact, we may define the prepotential for local mirror symmetry given in [CKYZ] in terms of the symplectic form (6.8).

For simplicity, let us assume $\dim H_c^2(X, \mathbf{Q}) = \dim H_c^4(X, \mathbf{Q})$ (, i.e. $r = s$), and thus (C_{ab}) is a square matrix. Also we assume $\det(C_{ab}) \neq 0$. In this case, inverting the matrix (C_{ab}), we may define the 'symplectic duals' of $\mathcal{O}_{C_1}(-J_1), \cdots, \mathcal{O}_{C_r}(-J_r)$

by $\sum_b C^{ab}\mathcal{O}_{D_b}(\mathcal{L}_b)$ $(a = 1,\cdots,r)$ where $(C^{ab}) := (C_{ab})^{-1}$. We should note that these symplectic duals are not in $K^c(X)$ in general, but are elements in $K^c(X)\otimes\mathbf{Q}$. Correspondingly we consider the symplectic duals of the mirror cycles $mir(\mathcal{O}_{C_a}(-J_a))$ by $\sum_b C^{ab}mir(\mathcal{O}_{D_b}(\mathcal{L}_b))$. From the period integrals of these cycles, we define the prepotential $F(t)$ by the special Kähler geometry relation;

$$(1\,,\,t_a\,,\,\frac{\partial F}{\partial t_a}\,) = (\,1\,,\,w_a^{(1)}(x)\,,\,\sum_b C^{ab}w_b^{(2)}(x)\,)\;, \tag{6.9}$$

where we use the fact $w^{(0)}(x) = Z(\mathcal{O}_p)\equiv 1$ for local mirror symmetry of $X=\widehat{\mathbf{C}^3/G}$ (cf. section 5). Integrating this special Kähler geometry relation, we obtain the prepotential, which gives the right predictions for Gromov-Witten invariants(, see [CKYZ] for examples).

For general G, the matrix (C_{ab}) may not be square. Even if it is square it might be singular. In such cases, some of the sheaves $\mathcal{O}_{C_a}(-J_a)$ do not have its symplectic duals. However the special Kähler relation (6.9) still makes sense under suitable modifications.

(6-4) Examples. Here we will present two examples to verify Conjecture 6.3. Some detailed physical analysis of D-branes and Π stability on the first example may be found in [DFR] (, see also [Ma][Do] and references therein). The hypergeometric series (with a different definition) of the second example are also studied in [dOFS] (see also [MR]).

Example (1): As the first example, we consider the case $X=\widehat{\mathbf{C}^3/\mathbf{Z}_3}$ with the group action specified by the weights $\frac{1}{3}(1,1,1)$. The toric resolution is the same as the G-Hilb and may be identified with the total space of the canonical bundle $K_{\mathbf{P}^2}$. The cohomology $H^{even}(X,\mathbf{Q})$ is generated by $1, J, J^2$ where J is the class dual to a line C in $\mathbf{P}^2$. As shown in the toric diagram in Fig.3, the toric divisor D represents the section $\mathbf{P}^2$. The construction of the secondary polytope is easy, and we find $l^{(1)} = (1,1,1,-3)$ for the local parameter $x=a^{l^{(1)}}$ of the GKZ hypergeometric system. With these data, we may write down the cohomology-valued hypergeometric series,

$$w(x;\frac{J}{2\pi i}) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(1+n+\rho)^3\Gamma(1-3(n+\rho))}x^{n+\rho}|_{\rho=\frac{J}{2\pi i}}. \tag{6.10}$$

The Taylor series expansion with respect to ρ and the ring structure of $H^{even}(X,\mathbf{Q})$ define the hypergeometric series of our interest. As conjectured in Conjecture 6.3, the integral and symplectic structure of the hypergeometric series comes from $K(X)$ and $K^c(X)$, respectively.

The tautological line bundles form an integral basis of $K(X)$ and, following [Re][Cr], they are easily determined as

$$\mathcal{F}_0 = \mathcal{O}_X\;,\;\mathcal{F}_1 = \mathcal{O}_X(J)\;,\;\mathcal{F}_2 = \mathcal{O}_X(2J)\;.$$

We can determine the dual basis S_k following [IN]. Here, instead, we fix the D-brane basis $\{\mathcal{B}_I\}$ to $\{\mathcal{O}_p,\mathcal{O}_C(-1),\mathcal{O}_D(-2)\}$, where $\mathcal{O}_C(-1):=\mathcal{O}_C\otimes\mathcal{O}_X(-J)$ and $\mathcal{O}_D(-2):=\mathcal{O}_D\otimes\mathcal{O}_X(-2J)$. The relations to the bases S_k are easily worked out to be

$$\mathcal{O}_p = S_0+S_1+S_2\;,\;\;\mathcal{O}_C(-1) = S_1+2S_2\;,\;\;\mathcal{O}_D(-2) = S_2, \tag{6.11}$$

and also the symplectic form (6.8) becomes

$$(\chi(\mathcal{B}_I,\mathcal{B}_J)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix} . \tag{6.12}$$

From the relations in (6.11), we find a basis $\{\mathcal{F}_0, \mathcal{F}_1 - \mathcal{F}_0, \mathcal{F}_0 - 2\mathcal{F}_1 + \mathcal{F}_2\}$ of $K(X)$, which are dual to $\{\mathcal{O}_p, \mathcal{O}_C(-1), \mathcal{O}_D(-2)\}$. Taking $ch : K(X) \to H^{even}(X, \mathbf{Q})$, we then finally arrive at a basis $\{Q_0, Q_2, Q_4\}$ of $H^{even}(X, \mathbf{Q})$,

$$Q_0 = ch(\mathcal{F}_0) \ \ , \ \ Q_2 = ch(\mathcal{F}_1) - ch(\mathcal{F}_0) \ \ , \ \ Q_0 = ch(\mathcal{F}_2) - 2ch(\mathcal{F}_1) + ch(\mathcal{F}_0) \ \ ,$$

in which we expand our cohomology-valued hypergeometric series,

$$w(x; \frac{J}{2\pi i}) = w^{(0)}(x)Q_0 + w^{(1)}(x)Q_2 + w^{(2)}(x)Q_4 \ \ .$$

For convenience, we write explicitly the coefficient hypergeometric series,

$$w^{(0)}(x) = 1 \ , \ w^{(1)}(x) = \partial_{\tilde{\rho}} w(x) \ , \ w^{(2)}(x) = \frac{1}{2}\partial_{\tilde{\rho}}^2 w(x) - \frac{1}{2}\partial_{\tilde{\rho}} w(x) \ , \tag{6.13}$$

where $\partial_{\tilde{\rho}} := \frac{1}{2\pi i}\frac{\partial}{\partial \rho}$.

Our Conjecture 6.3 predicts that the coefficient hypergeometric series $w^{(0)}(x)$, $w^{(1)}(x)$, $w^{(2)}(x)$ thus obtained have integral and symplectic monodromy properties with respect to the symplectic form (6.12). We verify these properties in Appendix A. Moreover our Conjecture 6.3 predicts that these hypergeometric series represents period integrals over the respective mirror cycles of $\mathcal{O}_p, \mathcal{O}_C(-1), \mathcal{O}_D(-2)$ under $mir : K^c(X) \xrightarrow{\sim} H_3(Y, \mathbf{Z})$, i.e.,

$$\begin{aligned} & w^{(0)}(x) = \int_{mir(\mathcal{O}_p)} \Omega(Y_x) \ \ , \\ & w^{(1)}(x) = \int_{mir(\mathcal{O}_C(-1))} \Omega(Y_x) \ \ , \ \ w^{(2)}(x) = \int_{mir(\mathcal{O}_D(-2))} \Omega(Y_x) \ \ , \end{aligned} \tag{6.14}$$

where Y is the mirror hypersurface $F(U, V, W_1, W_2) = 0$ in $\mathbf{C}^2 \times (\mathbf{C}^*)^2$ defined in (6.1). In Appendix A, we also construct explicitly the cycles which represent $mir(\mathcal{O}_p), mir(\mathcal{O}_C(-1))$ and $mir(\mathcal{O}_D(-2))$.

We remark that, in the case of the two dimensional singularity, the hyperkähler rotation helped us identify the mirror cycles in $H_2(Y, \mathbf{Z})$. In the present case, however, it is not so obvious to see that there is a canonical (geometric) way to construct the isomorphism $mir : K^c(X) \xrightarrow{\sim} H_3(Y, \mathbf{Z})$. Nevertheless, our Conjecture 6.3 predicts that there is in fact a canonical isomorphism encoded indirectly in our cohomology-valued hypergeometric series. This situation is parallel to the mirror symmetry of compact Calabi-Yau manifolds, where Strominger-Yau-Zaslow construction provides a recipe for the canonical mirror correspondence.

Finally we determine the prepotential. For this, we construct the symplectic dual to $\mathcal{O}_C(-1)$ by $-\frac{1}{3}\mathcal{O}_D(-2)$. Then from (6.14) the special Kähler relation becomes

$$(1, t, \frac{\partial F}{\partial t}) = (1, w^{(1)}(x), -\frac{1}{3}w^{(2)}).$$

Integrating this relation, we obtain the Gromov-Witten invariants of the geometry $X = K_{\mathbf{P}^2}$ (, the first column of Table 1 below). It should be noted that the symplectic dual $\frac{1}{3}\mathcal{O}_D(-2)$ is no longer in $K^c(X)$ but in $K^c(X) \otimes \mathbf{Q}$.

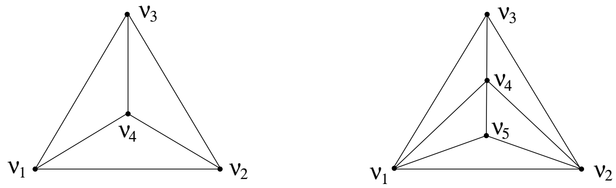

Fig.3. *The resolution diagrams of the singularities* $\mathbf{C}^3/G$ *with* $G = \langle \frac{1}{3}(1,1,1)\rangle = \mathbf{Z}_3$ *(left) and* $G = \langle \frac{1}{5}(1,1,3)\rangle = \mathbf{Z}_5$ *(right).*

Example (2): As the second example, we briefly sketch the case $X = \widehat{\mathbf{C}^3/\mathbf{Z}_5}$ with the group action specified by the weights $\frac{1}{5}(1,1,3)$. The crepant resolution is described in Fig.3. The toric divisors $D_4 := D_{\nu_4}$ and $D_5 := D_{\nu_5}$ are identified, respectively, with the Hirzebruch surface $\mathbf{F}_3$ and $\mathbf{P}^2$. We will denote the rational curves that appear as the zero section of $\mathbf{F}_3$ by $b \equiv C_1$, and also the fiber by $f \equiv C_2$. Thus the intersection numbers on $\mathbf{F}_3$ are $b \cdot b = -3$, $b \cdot f = 1$, $f \cdot f = 0$. $H^2(X, \mathbf{Z})$ is generated by J_1 and J_2 which measure the volumes $C_1 = b$ and $C_2 = f$, respectively (, i.e. $\int_{C_i} J_k = \delta_{ik}$). By the standard toric method (see, e.g., [Oda2][Fu]), we can determine the ring structure of the cohomology as

$$H^{even}(X, \mathbf{Q}) = \mathbf{Q}[J_1, J_2]/(J_1^2, J_2^2 - 3J_1J_2, J_2^3).$$

The construction of the secondary fan $Sec(\Sigma_G)$ is straightforward and we find $l^{(1)} = (1,1,0,1,-3)$ and $l^{(2)} = (0,0,1,-2,1)$ for the local parameters $x_1 = x := a^{l^{(1)}}$ and $x_2 = y := a^{l^{(2)}}$ of the hypergeometric series. With these data we can write down the cohomology-valued hypergeometric series (6.4) with

$$c(n,m) = \frac{1}{\Gamma(1+n)^2\Gamma(1+m)\Gamma(1+n-2m)\Gamma(1-3n+m)} \quad .$$

For the expansion of the cohomology-valued hypergeometric series, we first fix a symplectic D-brane basis of $K^c(X)$ by $\{\mathcal{B}_J\} = \{\mathcal{O}_p, \mathcal{O}_b(-J_1), \mathcal{O}_f(-J_2), \mathcal{O}_{\mathbf{P}^2}(-2J_1), \mathcal{O}_{\mathbf{F}_3}(-2J_2)\}$. Then it is straightforward to express these bases in terms of S_k by evaluating the pairings of $\mathcal{B}_J$'s with the tautological bundles [Re][Cr];

$$\mathcal{F}_0 = \mathcal{O}_X \ , \ \mathcal{F}_1 = \mathcal{O}_X(J_1) \ , \ \mathcal{F}_2 = \mathcal{O}_X(2J_1) \ , \ \mathcal{F}_3 = \mathcal{O}_X(J_2) \ , \ \mathcal{F}_4 = \mathcal{O}_X(J_1 + J_2) \ .$$

Here we summarize the results together with their duals under the pairing $K^c(X) \times K(X) \to \mathbf{Z}$:

$$\text{(6.15)} \qquad \begin{array}{rll} & K^c(X) & K(X) \\ \mathcal{O}_p & = S_0 + S_1 + S_2 + S_3 + S_4 & -\mathcal{F}_0 + 2\mathcal{F}_1 + 2\mathcal{F}_3 - 2\mathcal{F}_4 \\ \mathcal{O}_b(-J_1) & = S_1 + 2S_2 + S_4 & -\mathcal{F}_3 + \mathcal{F}_4 \\ \mathcal{O}_f(-J_2) & = S_3 + S_4 & \mathcal{F}_0 - 2\mathcal{F}_1 - \mathcal{F}_3 + 2\mathcal{F}_4 \\ \mathcal{O}_{\mathbf{P}^2}(-2J_1) & = S_2 & \mathcal{F}_0 - 2\mathcal{F}_1 + \mathcal{F}_2 \\ \mathcal{O}_{\mathbf{F}_3}(-2J_2) & = 2S_0 + S_1 & \mathcal{F}_0 - \mathcal{F}_1 - \mathcal{F}_3 + \mathcal{F}_4 \end{array}$$

The symplectic form on $K^c(X)$ may be evaluated using [Corollary 5.3. in IN] as

$$\text{(6.16)} \qquad (\chi(\mathcal{B}_I, \mathcal{B}_J)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \end{pmatrix} .$$

Evaluating the Chern characters of the tautological bundles in (6.15), we have for the basis $\{Q_J\}$ of $H^{even}(X,\mathbf{Q})$,

$$Q_0 = 1-2J_1J_2 \; ; \; Q_2^1 = J_1+\frac{1}{2}J_1^2+J_1J_2 \; , \; Q_2^2 = J_2+\frac{7}{2}J_1J_2 \; ; \; Q_4^1 = J_1^2 \; , \; Q_4^2 = J_1J_2 \; ,$$

by which we arrange the cohomology-valued hypergeometric series as

$$w(x;\frac{J}{2\pi i}) = w^{(0)}(x)Q_0 + \sum_{a=1}^{2} w_a^{(1)}(x)Q_2^a + \sum_{b=1}^{2} w_b^{(2)}(x)Q_4^b \; ,$$

(where $w^{(0)}(x) \equiv 1$ from the form $c(n,m)$ above). Our Conjecture 6.3 predicts that the coefficient hypergeometric series $\{w^{(0)}(x), w_a^{(1)}(x), w_b^{(2)}(x)\}$ should be integral and symplectic with respect to the symplectic form (6.16).

As a check of the mirror symmetry, we determine the prepotential. First, we write the special Kähler relation (6.9),

$$(\, 1 \, , \, t_a \, , \, \frac{\partial F}{\partial t_a} \,) = (\, 1 \, , \, w_a^{(1)}(x) \, , \, \sum_b C^{ab} w_b^{(2)}(x) \,) \; ,$$

with the matrix $(C^{ab}) = \left(\begin{smallmatrix} -3 & 1 \\ 1 & -2 \end{smallmatrix}\right)^{-1}$. From this relation, it is straightforward to obtain

$$\begin{aligned} \frac{\partial F}{\partial t_1} &= -\frac{1}{5}(\partial_{\tilde\rho_1}^2 + \partial_{\tilde\rho_1}\partial_{\tilde\rho_2} + \frac{3}{2}\partial_{\tilde\rho_2}^2)w(x) + \frac{2}{5}t_1 + \frac{7}{10}t_2 - \frac{2}{5} \;\; , \\ \frac{\partial F}{\partial t_2} &= -\frac{1}{10}(\partial_{\tilde\rho_1}^2 + 6\partial_{\tilde\rho_1}\partial_{\tilde\rho_2} + 9\partial_{\tilde\rho_2}^2)w(x) + \frac{7}{10}t_1 + \frac{27}{10}t_2 - \frac{6}{5} \;\; , \end{aligned}$$

where $\partial_{\tilde\rho_k} = \frac{1}{2\pi i}\partial_{\rho_k}$. Just as the case for compact Calabi-Yau manifolds, we can integrate the special Kähler geometry relation with the prepotential

$$F(t) = \frac{1}{6}(-\frac{2}{5}t_1^3 - \frac{3}{5}t_1^2t_2 - \frac{9}{5}t_1t_2^2 - \frac{9}{5}t_2^3) + \frac{7}{10}t_1t_2 + \frac{1}{5}t_1^2 + \frac{27}{20}t_2^2 - \frac{2}{5}t_1 - \frac{6}{5}t_2 + F_{inst}(t) \; ,$$

where $F_{inst}(t) := \sum_{n,m\geq 0} N(n,m)q_1^n q_2^m$ represents the quantum corrections. Here $N(n,m)$ are the genus zero Gromov-Witten invariants of the homology class $\beta = nb+mf \in H_2(X,\mathbf{Z})$, i.e. $N(n,m) = N_0(nb+mf)$ in the standard notation $N_g(\beta)$ of Gromov-Witten invariants of genus g in literatures. In Table 1, we have listed the integral 'invariants' called Gopakumar-Vafa invariants $\tilde N(n,m)$ [GV]. One can verify that Gromov-Witten invariants coincides with the results from topological vertex technique (see [AKMV]).

We may observe the similar integrability of the special Kähler relation for many other examples. In compact situations, this integrability (or more specifically the existence of the so-called Griffiths-Yukawa coupling) is a consequence of the Griffiths transversality (see [St]) for which we use an integration over (mirror) manifolds. However it should be noted that the existence of the prepotential for non-compact cases seems non-trivial, since integrating over a non-compact manifolds needs some special care.

n\m	0	1	2	3	4	5	6
0	0	-2	0	0	0	0	0
1	3	4	3	5	7	9	11
2	-6	-10	-12	-12	-24	-56	-140
3	27	64	91	108	150	294	675
4	-192	-572	-980	-1332	-1808	-2982	-5992
5	1695	6076	12259	18912	26983	42005	76608
6	-17064	-71740	-166720	-289440	-443394	-689520	-1192644
7	188454	909760	2394779	4632120	7665776	12254816	20764870

Table 1. Gopakumar-Vafa numbers $\tilde{N}_0(n,m) = \tilde{N}_0(nb+mf)$ for $X = \widehat{\mathbf{C}^3/\mathbf{Z}_5}$. $\tilde{N}_0(n,0)$ coincides with the Gopakumar-Vafa numbers for $\widehat{\mathbf{C}^3/\mathbf{Z}_3}$.

7. **Conclusion and discussions**

We have studied in detail the cohomology-valued hypergeometric series for the case of non-compact Calabi-Yau manifolds. Giving an interpretation of $w(x; \frac{J}{2\pi i})$ as the central charge formula (Conjecture 6.3), we have provided supporting evidence for the conjecture in the cases of $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$, and also $X = G$-Hilb $\mathbf{C}^3$ with finite abelian subgroups $G = \mathbf{Z}_3, \mathbf{Z}_5 \subset SL(3,\mathbf{C})$. We have also clarified the structure of the prepotential for the non-compact Calabi-Yau geometries $X = G$-Hilb $\mathbf{C}^3$.

As a byproduct of our study, we have found that K.Saito's system of differential equations satisfied by the primitive forms may be replaced by a suitable (resonant and reducible) GKZ system, whose solutions we may set up easily.

As addressed after Conjecture 2.2, our cohomology-valued hypergeometric series (or the central charge formula) connects two different 'monodromy' properties, Fourier-Mukai transforms and the monodromy transforms of hypergeometric series (Dehn twists in the symplectic mapping class group). We see that the latter monodromy property arises associated with the discriminant locus in $\mathcal{M}_{Sec(\Sigma)}$. As shown in section 3 for $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$, the discriminant splits into several irreducible components in the q-coordinate and the monodromy transform around each irreducible component may be identified with a suitable twist functor. From these facts, it is conceivable that the group of self-equivalences of $D^b(Coh(X))$, i.e. Auteq $D^b(X)$, is generated by these 'monodromy' transformations up to the shift functors,

$$\mathrm{Auteq} D^b(\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}})/\{\ [k]\,|\, k\in\mathbf{Z}\ \} = \Big\langle R_1,\cdots,R_\mu, T_{\mathcal{O}_{C_1}(-1)},\cdots,T_{\mathcal{O}_{C_\mu}(-1)}\Big\rangle\ ,$$

where R_k represents the functor tensoring the tautological line bundle $\mathcal{F}_k$ and $T_{\mathcal{O}_{C_k}(-1)}$ is the Seidel-Thomas twist. (This been proved affirmatively in a recent paper by Ishii and Uehara [IU].) In Proposition 4.4, we have seen explicitly that these functors are represented by the corresponding actions on the roots $\beta_k(a)$'s, i.e., changing the phases of the roots and permuting (braiding) of the roots $\beta_k(a)$'s. This simple picture should provide us an intuition about the possible forms of the self-equivalences in $DFuk(Y,\beta)$.

A similar correspondence between monodromy transforms and Fourier-Mukai transforms, and also the canonical mirror isomorphism $mir : K^c(X) \xrightarrow{\sim} H_3(Y, \mathbf{Z})$ are contained in our Conjecture 6.3 for three dimensional cases $X = \widehat{\mathbf{C}^3/G} = G$-Hilb $\mathbf{C}^3$ [IN][BKR][CI]. In three dimensions, there can be several Calabi-Yau resolutions of the singularity $\mathbf{C}^3/G$, and correspondingly there are many different types of (Fourier-Mukai) transforms connecting these different resolutions [BKR][CI]. It is interesting to study the details of the canonical mirror isomorphism mir from our cohomology-valued hypergeometric series in such situations.

As we have done for $X = \widehat{\mathbf{C}^3/\mathbf{Z}_3}$, in some cases with lower $\dim H^2(X, \mathbf{Q})$, we can determine explicitly the integral cycles and their monodromy properties to provide a consistency check for our Conjecture 6.3. We can also read off the isomorphism $mir : K^c(X) \xrightarrow{\sim} H_3(Y, \mathbf{Z})$ from our cohomology-valued hypergeometric series. This isomorphism mir should also follow from the geometric framework like the Strominger-Yau-Zaslow construction, which provides a canonical equivalence $Mir : D^b_c(Coh(X)) \xrightarrow{\sim} DFuk(Y, \beta)$.

Finally, as for the compact Calabi-Yau (hypersurfaces), supporting evidence for our Conjecture 2.2 is reported in [Hos], especially for dimensions one and two. However, for example, the observation (1.4) made for quintic hypersurface X_5 still needs a justification. Namely we have to work out a symplectic basis of the K-group $K(X_5)$ which yields the specific forms of the 'charges' $1, J - \frac{c_1(X_5)J}{12} - \frac{11}{2}\frac{J^2}{5}, \frac{J^2}{5}, \frac{J^3}{5}$, and also their mirror cycles in $H_3(X_5^\vee, \mathbf{Z})$ with their period integrals $w^{(0)}(x), w^{(1)}(x), w^{(2)}(x), w^{(3)}(x)$. From the SYZ construction [SYZ][LYZ] we may expect that $K(X_5)$ is generated by the structure sheaf $\mathcal{O}_X$, a skyscraper sheaf $\mathcal{O}_p$, and additional sheaves $\mathcal{E}$ and $\mathcal{F}$ which, respectively, have their support on a divisor and a curve (,i.e. $D4$ and $D2$ branes). The cycles which give the period integrals $w^{(0)}(x)$ and $w^{(3)}(x)$ are known from the original work [CdOGP]. In fact, claiming these cycles to be the mirror images of $\mathcal{O}_p$ and $\mathcal{O}_X$, i.e. the Lagrangian torus cycle T^3 and the section S^3 of the fibration, respectively, was the starting point of the SYZ construction. However we still lack knowledge for the remaining cycles and also the corresponding sheaves $\mathcal{E}$ and $\mathcal{F}$.

Appendix A. Integrating over vanishing cycles

(A-1) $\mathbf{C}^2/\mathbf{Z}_{\mu+1}$. As a warm up, let us first evaluate the period integral (4.2) of the primitive form. We will obtain the central charge $Z_t(S_k)$ in (5.4) of the sheaf $S_k = \mathcal{O}_{C_i}(-1)$ as the integral over the vanishing cycle $L_i = mir(\mathcal{O}_{C_i}(-1))$.

As in Proposition 4.2, we write the roots of the polynomial $\psi(W) = a_0 + a_1 W + \cdots + a_{\mu+1}W^{\mu+1}$ by $\beta_0, \cdots, \beta_\mu$. Then for the defining equation of the singularity, we have

$$f(a,W) + U^2 + V^2 = (W-\beta_0)\cdots(W-\beta_\mu) + U^2 + V^2 \ ,$$

where we set $a_{\mu+1} = 1$ by scaling the variable W. Assume that all the roots β_i are real and $\beta_0 < \beta_1 < \cdots < \beta_\mu$. When $f(a,W) \leq 0$ holds for $\beta_{k-1} \leq W \leq \beta_k$, we may construct a vanishing cycle L_k as $L_k = L_k^+ \cup L_k^- \approx S^2$ with

$$L_k^{\pm} = \{ (\pm\sqrt{|f(a,W)| - V^2}, V, W) \in \mathbf{R}^3 \mid f_\Sigma(a,W) + V^2 \leq 0,\ \beta_{k-1} \leq W \leq \beta_k \ \} \, .$$

Similarly, when $f(a,W) \geq 0$ for $\beta_{k-1} \leq W \leq \beta_k$, we construct a vanishing cycle changing the variables; $U \to \sqrt{-1}U, V \to \sqrt{-1}V$. For the first case, the evaluation of the period integral over L_k^+ proceeds as

$$\begin{aligned}
&\int_{L_k^+} Res_{F_\Sigma=0}\left(\frac{dWdUdV}{f(W)+U^2+V^2}\right) = \int_{\beta_{k-1}}^{\beta_k} dW \int_{-\sqrt{|f(W)|}}^{\sqrt{|f(W)|}} \frac{(\frac{1}{2})\, dV}{\sqrt{|f(W)|-V^2}} \\
&= \frac{1}{2}\int_{\beta_{k-1}}^{\beta_k} dW\ \mathrm{Sin}^{-1}\left(\frac{V}{\sqrt{|f(W)|}}\right)\Bigg|_{-\sqrt{|f(W)|}}^{\sqrt{|f(W)|}} = \frac{\pi}{2}\int_{\beta_{k-1}}^{\beta_k} dW \ \ ,
\end{aligned} \tag{A.1}$$

where $F_\Sigma := f(W) + U^2 + V^2$. We have a similar result for L_k^-, and in total $\pi(\beta_k - \beta_{k-1})$ for the period integral. Since we have μ-independent vanishing cycles in this way, we come to the well-known results summarized in Proposition 4.1, and also Proposition 4.2. In the above evaluation, one should note that the period integrals of the primitive form measure the volumes of the vanishing cycles.

The period integral $\Pi_\gamma(a)$ (3.1) has a slightly different shape in the integration measure. First let us take $\gamma = L_k \times S^1$ for the cycle $\gamma \in H_3(\mathbf{C}^2 \times \mathbf{C}^* \setminus (F_\Sigma = 0), \mathbf{Z})$ with a loop S^1 which encircles the hypersurface. Since the integration over S^1 produces the holomorphic two form $\Omega(Y_x) = \frac{i}{2\pi^2} Res(\frac{1}{f(W)+U^2+V^2} dUdV\frac{dW}{W})$, we have

$$\int_{L_k^+} \Omega(Y_x) = \frac{i}{4\pi^2}\int_{\beta_{k-1}}^{\beta_k} \frac{dW}{W}\int_{-\sqrt{|f(W)|}}^{\sqrt{|f(W)|}} \frac{dV}{\sqrt{|f(W)|-V^2}} = \frac{i}{4\pi}\int_{\beta_{k-1}}^{\beta_k}\frac{dW}{W} \ \ , \tag{A.2}$$

and a similar result for L_k^-. Thus we arrive at the central charge of the sheaves $S_k = \mathcal{O}_{C_k}(-1)$ in (5.4) and justifies our claim that $mir(\mathcal{O}_{C_k}(-1)) = L_k$ that follows from the cohomology-valued hypergeometric series (5.3). One should note that the logarithmic behaviors of the solutions in Proposition 4.4 come from the torus invariant measure $\frac{dW}{W}$ in (3.1). The claimed relation $mir(\mathcal{O}_p) = T_0$ for a cycle $T_0 \approx S^1 \times S^1$ can also be justified by evaluating the period integral $\int_{T_0}\Omega(Y_x) = 1$ (, see the corresponding cycle T_0 and the evaluation of the period integral over it in three dimensions below).

(A-2) $\mathbf{C}^3/\mathbf{Z}_3$. The evaluations of the period integrals in (A.1),(A.2) extend to three dimensions. In three dimensions, however, we see that some vanishing cycles

require careful treatment and in fact become non-compact cycles with non-trivial fundamental groups. This non-compactness of certain (vanishing) cycles comes from the tori $(\mathbf{C}^*)^2$ of the ambient scape $\mathbf{C}^2 \times (\mathbf{C}^*)^2$, and is detected only by the invariant measure of the period integral of the local mirror symmetry (A.2). To illustrate this, we first construct all the cycles which reproduce the period integrals (6.14). We will then contrast the results to the corresponding period integrals of the primitive form.

- *(Non-compact) Vanishing cycles and period integrals.*

Let us start with the toric data for the crepant resolution $\widehat{\mathbf{C}^3/\mathbf{Z}^3}$, i.e., $\nu_1 = (1,0,0), \nu_2 = (0,1,0), \nu_3 = (0,0,1), \nu_4 = (1/3,1/3,1/3)$ in N_G. We fix an isomorphism $\varphi : N_G \cong \mathbf{Z}^3$ such that $\varphi(\nu_1) = (1,0,0), \varphi(\nu_2) = (1,2,1), \varphi(\nu_3) = (1,1,2)$, $\varphi(\nu_4) = (1,1,1)$, and consider the following deformation family of hypersurfaces in $\mathbf{C}^2 \times (\mathbf{C}^*)^2$;

$$F_\Sigma(a,U,V,W) := a_1 + a_2 W_1^2 W_2 + a_3 W_1 W_2^2 + a_4 W_1 W_2 + U^2 + V^2 = 0 \ ,$$

with $(a_1,..,a_4) \in (\mathbf{C}^*)^4$. The period integral of the local mirror symmetry is defined in general in (6.2) and satisfies the GKZ hypergeometric system (6.3) in Proposition 6.2. This GKZ hypergeometric system introduces the toric compactification $\mathcal{M}_{Sec(\Sigma)}$ in terms of the secondary fan $Sec(\Sigma)$ for the deformation parameters $(a_1,..,a_4)$. In the present case, we have $\mathcal{M}_{Sec(\Sigma)} \cong \mathbf{P}^1$ with the local parameter $x = \frac{a_1 a_2 a_3}{a_4^3}$ near the large complex structure limit. By scaling the variables, we may set $(a_1,a_2,a_3,a_4) = (a,1,1,1)$, and hence we have the mirror family $\{Y_x\}$ with the period integral parametrized by $x = a$;

$$\text{(A.3)} \qquad \int_L \Omega(Y_x) = \frac{1}{4\pi^3} \int_L Res_{F_\Sigma=0} \left(\frac{1}{a + f_1(W) + U^2 + V^2} dU dV \frac{dW_1}{W_1} \frac{dW_2}{W_2} \right) \ ,$$

where $a + f_1(W) = a + W_1 W_2 (1 + W_1 + W_2)$ and L is a cycle in $H_3(Y_x, \mathbf{Z})$. In what follows, we construct integral cycles L and evaluate the above period integral.

First, we construct a torus cycle $T_0 \approx S^1 \times S^1 \times S^1$ which fixes the normalization of the period integral. To do this, we introduce new variables $\tilde{U} = U + iV, \tilde{V} = U - iV$ and write the defining equation as

$$F_\Sigma(a,U,V,W) = a + f_1(W) + U^2 + V^2 = a + f_1(W) + \tilde{U}\tilde{V} \ .$$

We define the cycle to be

$$T_0 := \{(\tilde{U},\tilde{V},W_1,W_2) \in \mathbf{C}^2 \times (\mathbf{C}^*)^2 \mid \tilde{U} = \frac{-a - f_1(W)}{\tilde{V}}, \ |\tilde{V}| = |W_1| = |W_2| = \varepsilon\} \ .$$

Then it is straightforward to evaluate $\int_{T_0} \Omega(Y_x) = 1$ since

$$\frac{1}{4\pi^3} Res_{F_\Sigma=0} \left(\frac{(-2i)^{-1} d\tilde{U} d\tilde{V}}{a + f_1(W) + \tilde{U}\tilde{V}} \frac{dW_1}{W_1} \frac{dW_2}{W_2} \right) = \frac{1}{(2\pi i)^3} \frac{d\tilde{V}}{\tilde{V}} \frac{dW_1}{W_1} \frac{dW_2}{W_2} \ .$$

This normalizes the period integral and verifies the first relation in (6.14) with $mir(\mathcal{O}_p) = T_0$.

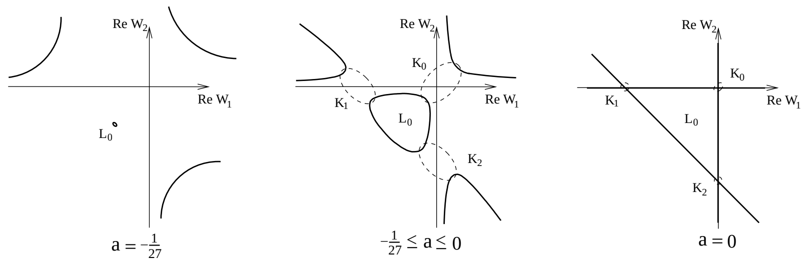

Fig.4. *Vanishing cycles near the critical points* $(0,0),(-1,0),(0,-1),(-\frac{1}{3},-\frac{1}{3})$ *of the hypersurface* $a+W_1W_2(1+W_1+W_2)+U^2+V^2=0$ *in* $\mathbf{C}^2\times(\mathbf{C}^*)^2$ $(-\frac{1}{27}\le a\le 0)$*. They are termed* K_0,K_1,K_2 *and* L_0*, respectively, and have topologies* $K_i\approx S^3\setminus S^1\,(i=0,1,2)$ *and* $L_0\approx S^3$*.*

Secondly, we describe a vanishing cycle which verifies the third relation in (6.14). Following general arguments on singularities, e.g. in [AGV], we may have vanishing cycles looking at the critical points of $f_1(W)$(, more precisely $f_1(W)+U^2+V^2$). We see that $f_1(W)=W_1W_2(1+W_1+W_2)$ has four critical points,

$$(W_1,W_2)_{crit}=(0,0),(-1,0),(0,-1),(-1/3,-1/3),$$

with critical values $0,0,0,\frac{1}{27}$, respectively. The vanishing cycle at $(-1/3,-1/3)$ is easy to describe in a way similar to (A-1). The result is $L_0:=L_0^+\cup L_0^-\approx S^3$ with

$$L_0^{\pm}=\left\{(\sqrt{-1}u,\sqrt{-1}v,W_1,W_2)\;\middle|\;\begin{matrix}v=\pm\sqrt{a+f_1(W)-u^2},\;\;a+f_1(W)\ge 0,\\ -\sqrt{a+f_1(W)}\le u\le\sqrt{a+f_1(W)}\end{matrix}\right\},$$

where u,v,W_1,W_2 above are supposed to be real, and also $-\frac{1}{27}\le a\le 0$ (see Fig.4). The evaluation of the period integral is quite parallel to (A.2), and we have

$$\text{(A.4)}\qquad \int_{L_0}\Omega(Y_x)=\frac{1}{4\pi^2}\int_{a+W_1W_2(1+W_1+W_2)\ge 0}\frac{dW_1}{W_1}\frac{dW_2}{W_2}\,,$$

where $\int_{a+W_1W_2(1+W_1+W_2)\ge 0}$ means the integral over the closed domain which appears in the third quadrant of the real W_1,W_2 plane for $-\frac{1}{27}\le a\le 0$ (see Fig.4).

We may evaluate the integral (A.4) near $a=0$ in a straightforward way, by setting $X=W_1,Y=W_2$, as

$$\frac{1}{4\pi^2}\int_{Y_{min}(a)}^{Y_{max}(a)}\frac{dY}{Y}\int_{X_{min}(a,Y)}^{X_{max}(a,Y)}\frac{dX}{X}=\frac{1}{8\pi^2}(\log(-a))^2+\cdots,$$

where $X_{min}(a,Y)<X_{max}(a,Y)$ are roots of $a+XY(1+X+Y)=0$, and $Y_{min}(a)$, $Y_{max}(a)$ are determined from the conditions

$$a+XY(1+X+Y)=\frac{\partial}{\partial X}(a+XY(1+X+Y))=0$$

and

$$Y_{min}(-\frac{1}{27})=Y_{max}(-\frac{1}{27})=-1/3.$$

The behavior $\frac{1}{8\pi^2}(\log(-a))^2+\cdots$ is easily seen by observing

$$\int_{X_{min}(a,Y)}^{X_{max}(a,Y)}\frac{dX}{X}=\log(-a)-\log(-Y(1+Y)^2)+O(a)$$

and

$$Y_{min}(a)=-1-2\sqrt{-a}-2a+\cdots,\;Y_{max}(a)=4a-32a^2+\cdots$$

for small $|a|$. Clearly this period integral vanishes at $a=-\frac{1}{27}$. This vanishing property and the $(\log(-a))^2$ behavior near $a=0$ suffice to identify the period integral with $w^{(2)}(x)=\frac{1}{2}\partial_{\tilde\rho}^2 w(x)-\frac{1}{2}\partial_{\tilde\rho}w(x)$ in (6.13), which has the same vanishing property at $x=-\frac{1}{27}$ and also the expansion near $x=0$,

$$w^{(2)}(x)=-\frac{1}{8\pi^2}(\log(-x))^2+\frac{1}{8}+\frac{3}{2\pi^2}x\log(-x)+\frac{9}{4\pi^2}x-\frac{423}{16\pi^2}x^2+\cdots\ .$$

The difference in sign is simply a matter of the orientation of the vanishing cycle L_0. Thus we verify the third relation in (6.14) with $mir(\mathcal{O}_D(-2))=L_0\approx S^3$ up to the orientation of the cycle.

Finally, let us look carefully at the vanishing cycles at the critical points $(0,0)$, $(-1,0)$, $(0,-1)$, which we call K_0,K_1,K_2, respectively. Usually vanishing cycles have topology of S^3. However this is not the case for the cycles K_i's since we consider the mirror geometry in $\mathbf{C}^2\times(\mathbf{C}^*)^2$, (see Fig 4). It will turn out that they have topology $S^3\setminus S^1$ and hence non-compact.

Let us first describe the vanishing cycle at the critical point $(0,0)$. Similarly to L_0, we define $K_0=K_0^+\cup K_0^-$ using real variables A,B,U,V by

$$K_0^{\pm}=\left\{(U,V,A+iB,A-iB)\,\middle|\,\begin{array}{c}U=\pm\sqrt{|a+f_1(A,B)|-V^2},\ \ a+f_1(A,B)\le 0,\\ -\sqrt{|a+f_1(A,B)|}\le V\le\sqrt{|a+f_1(A,B)|}\end{array}\right\},$$

where $a+f_1(A,B)=a+(A^2+B^2)(1+2A)$. Note that the locus $A=B=0$ is not contained in $K_0^{\pm}$ because $W_1=A+iB, W_2=A-iB$ live in $\mathbf{C}^*$. Since the excluded loci $A=B=0$ in K_0^+ and K_0^- make up S^1, we see that $K_0=K_0^+\cup K_0^-\approx S^3\setminus S^1$. After some calculations similar to (A.2), we obtain

$$\int_{K_0}\Omega(Y_x)=\frac{1}{4\pi^3}\pi\int_{a+(A^2+B^2)(1+2A)\le 0}\frac{-2idAdB}{(A^2+B^2)}\ ,$$

where we mean by $a+(A^2+B^2)(1+2A)\le 0$ the closed domain which appears near the origin in the real A,B plane for $-\frac{1}{27}\le a\le 0$. Note that the integral diverges if the locus $A=B=0$ were contained in K_0. We may isolate this divergent contribution from the origin by introducing the polar coordinate $r\cos\theta=\sqrt{1+2A}\,A$, $r\sin\theta=\sqrt{1+2A}\,B$ satisfying $r^2\cos^2\theta=(1+2A)A^2$. The calculation proceeds as follows;

$$\begin{aligned}\int_{K_0}\Omega(Y_x)&=\frac{-i}{2\pi^2}\int_{\sqrt{\epsilon}}^{\sqrt{-a}}\frac{dr}{r}\int_0^{2\pi}d\theta\frac{1+2A(r,\theta)}{1+3A(r,\theta)}\\&=\frac{1}{2\pi i}\frac{1}{3}(-6a+45a-560a^3+\cdots)+\frac{1}{2\pi i}(\log(-a)-\log\epsilon)\ ,\end{aligned}\tag{A.5}$$

where $\epsilon>0$ is a small constant to handle the noncompact cycle $K_0\approx S^3\setminus S^1$, and the divergent term $\log\epsilon$ when $\epsilon\to 0$ will be ignored in the following. (Here is an ambiguity about the cut of the function log. Precisely, we should say that we drop the divergent term $\log(-\epsilon)=\log\epsilon-\pi i$.) This series expansion should be compared with the hypergeometric series $w^{(1)}(x)=\partial_{\tilde\rho}w(x)$ in (6.13), which has the form near $x=0\,(x=a)$

$$w^{(1)}(x)=\frac{1}{2\pi i}\log(x)+\frac{1}{2\pi i}(-6x+45x^2-560x^3+\frac{17325}{2}x^4+\cdots)\ .$$

The slight difference in the factor $\frac{1}{3}$ can be explained by the contribution from the similar non-compact vanishing cycles K_1,K_2 at the critical points $(-1,0)$ and

$(0,-1)$, respectively. Let us describe the cycle $K_1 = K_1^+ \cup K_1^-$ by

$$K_1^{\pm} = \left\{ (U, V, -1-2A, A+iB) \,\middle|\, \begin{array}{c} U = \pm\sqrt{|a + f_1(A,B) - V^2|}, \;\; a + f_1(A,B) \leq 0, \\ -\sqrt{|a+f_1(A,B)|} \leq V \leq \sqrt{|a+f_1(A,B)|} \end{array} \right\}.$$

Here similarly to K_0, $a + f_1(A,B) = a + (A^2+B^2)(1+2A)$ and we mean by $a + f_1(A,B) \leq 0$ to represent the closed domain near the origin in the real A, B plane which appears for $-\frac{1}{27} \leq a \leq 0$. As is the case for K_0, the locus $A = B = 0$ is excluded and hence we have $K_1 \approx S^3 \setminus S^1$. The integration over this non-compact cycle K_1 may be done by changing variables to the polar coordinates. After some manipulations, we get a similar result to (A.5);

$$\begin{aligned} \int_{K_0} \Omega(Y_x) &= \frac{-i}{2\pi^2} \int_{\sqrt{\epsilon}}^{\sqrt{-a}} \frac{dr}{r} \int_0^{2\pi} d\theta \frac{-A(r,\theta)}{1+3A(r,\theta)} \\ &= \frac{1}{2\pi i} \frac{1}{3} (-6a + 45a - 560a^3 + \cdots) \,, \end{aligned} \tag{A.6}$$

which does not contain the divergent term $\log(-a) - \log\epsilon$. We should note in the difference between (A.5) and (A.6) that K_0 is *not* homologous to K_1 in $H_3(Y_x, \mathbf{Z})$ although they have the same topology $S^3 \setminus S^1$.

By symmetry under $W_1 \leftrightarrow W_2$, we have a similar construction of the cycle $K_2 = K_2^+ \cup K_2^- \approx S^3 \setminus S^1$ and obtain the same result as above for $\int_{K_2} \Omega(Y_x)$. Putting all the above results together, we obtain $\int_{K_0+K_1+K_2} \Omega(Y_x) = w^{(1)}(x)$, which entails $mir(\mathcal{O}_C(-1)) = K_0 + K_1 + K_2$ from (6.14).

To summarize, we read the mirror isomorphism $mir : K^c(X) \xrightarrow{\sim} H_3(Y, \mathbf{Z})$ from the first part of our Conjecture 6.3 (and the equation (6.14)), up to orientations of cycles, as

$$mir(\mathcal{O}_p) = T_0 \;\;,\;\; mir(\mathcal{O}_C(-1)) = K_0 + K_1 + K_2 \;\;,\;\; mir(\mathcal{O}_D(-2)) = L_0 \,, \tag{A.7}$$

where the topology of the cycles are $T_0 \approx S^1 \times S^1 \times S^1$, $K_i \approx S^3 \setminus S^1 (i = 0,1,2)$ and $L_0 \approx S^3$.

• *Monodromy and the symplectic form.*

We have identified the period integrals over the cycles with the hypergeometric series by $\int_{T_0} \Omega(Y_x) = w^{(0)}(x)$, $\int_{K_0+K_1+K_2} \Omega(Y_x) = w^{(1)}(x)$, $\int_{L_0} \Omega(Y_x) = w^{(2)}(x)$, where $(w^{(0)}(x), w^{(1)}(x), w^{(2)}(x)) = (1, \partial_{\tilde\rho} w(x), \frac{1}{2}\partial_{\tilde\rho}^2 w(x) - \frac{1}{2}\partial_{\tilde\rho} w(x))$. The calculation of the monodromy of the hypergeometric series ${}^t(w^{(0)}(x), w^{(1)}(x), w^{(2)}(x))$ is straightforward (see e.g. [Hos]), and we obtain the monodromy matrices about $x = 0, x = -\frac{1}{27}, x = \infty$, respectively, as

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \;,\; M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \;,\; M_\infty = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \,,$$

where $M_\infty := (M_1 M_0)^{-1}$. Now we may verify the second part of Conjecture 6.3 by observing the equality of the two symplectic forms,

$$(\chi(\mathcal{B}_I, \mathcal{B}_J)) = (\# L_I \cap L_J) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix} \,,$$

for the basis $\{\mathcal{B}_I\} = \{\mathcal{O}_p, \mathcal{O}_C(-1), \mathcal{O}_D(-2)\}$ of $K^c(X)$ and its mirror image $\{L_I\} = \{T_0, K_0 + K_1 + K_2, L_0\}$ in $H_3(Y, \mathbf{Z})$. The intersection numbers among the cycles

L_I's are easily determined from their definitions(, see also Fig.4,) under suitable orientations. Now the second part of Conjecture 6.3 follows from the Picard-Lefschetz theory of the cycles L_I. By the mirror isomorphism $mir : K^c(X) \tilde{\rightarrow} H_3(Y,\mathbf{Z})$, it is conjectured that the cycles $T_0, K_0+K_1+K_2, L_0$ constitute an integral basis of $H_3(Y,\mathbf{Z})$.

• *Period integral of the primitive form.*

Here to have things to be contrasted, let us evaluate the period integrals of the primitive form,

$$\int_L \mathcal{U}(a) = \int_L Res_{F_\Sigma=0}\left(\frac{dUdVdW_1dW_2}{a+f_1(W)+U^2+V^2}\right) \tag{A.8}$$

for the cycles $L = T_0, K_0, K_1, K_2, L_0$ which are considered in the hypersurface $a+f_1(W)+U^2+V^2=0$ in $\mathbf{C}^4$. Firstly, it is rather clear that the homology class T_0 is trivial and to have $\int_{T_0}\mathcal{U}(a)=0$. Also all the cycles K_0, K_1, K_2 and L_0 are simply vanishing cycles having topology of S^3. The evaluations of the period integral are similar to the previous case of the local mirror symmetry, and it is straightforward to obtain

$$\int_{L_0}\mathcal{U}(a) = \pi\int_{a+f_1(W)\geq 0} dW_1dW_2 = \frac{2\pi^2}{9\sqrt{3}}(1+27a)\,{}_2F_1(\frac{1}{3},\frac{2}{3},2,1+27a),$$

where $f_1(W)=W_1W_2(1+W_1+W_2)$ and, as before, $a+f_1(W)\geq 0$ represents the closed domain in the third quadrant in the real W_1, W_2 plane for $-\frac{1}{27}\leq a\leq 0$. For the cycles K_j $(i=0,1,2)$, we have the same results,

$$\int_{K_j}\mathcal{U}(a) = (-2i)\pi\int_{a+f_1(A,B)\leq 0} dAdB = 2\pi^2 i\; a\,{}_2F_1(\frac{1}{3},\frac{2}{3},2,-27a)\ \ ,$$

where $f_1(A,B)=(A^2+B^2)(1+2A)$ and $a+f_1(A,B)\leq 0$ represents a closed domain near the origin of the real A,B plane which exists for $-\frac{1}{27}\leq a\leq 0$. The result above indicates that the cycles K_0, K_1, K_2 are homologically equivalent. In fact, we may verify the following Picard-Fuchs equation of second order satisfied by the period integral (A.8);

$$\{\theta_a(\theta_a-1)+3a(3\theta_a-2)(3\theta_a-1)\}\int_L\mathcal{U}(a)=0\ \ , \tag{A.9}$$

where $\theta_a = a\frac{\partial}{\partial a}$. This equation indicates that there exit only two homologically independent cycles. In appendix B, we will derive this Picard-Fuchs equation starting from the GKZ system appearing in Remark after Proposition 6.2. This equation should be compared with the differential equation

$$\{\theta_x^2+3x(3\theta_x+2)(3\theta_x+1)\}\theta_x\int_L\Omega(Y_x)=0\ \ , \tag{A.10}$$

where $\theta_x = x\frac{\partial}{\partial x}$. The latter third order differential equation, which follows also from the GKZ system in Proposition 6.2, characterizes the period integrals over the cycles (A.7).

Appendix B. **Differential equation (A.9)**

Here we derive the second order differential equation (A.9) form the GKZ system for the primitive form (cf.(6.3)) observing a factorization of a first order differential operator. The same second order differential equation can be derived from the K.Saito's system.

As in (A-2), we fix an isomorphism $\varphi : N_G \xrightarrow{\sim} \mathbf{Z}^3$ so that

$$(\varphi(\nu_1)\ \varphi(\nu_2)\ \varphi(\nu_3)\ \varphi(\nu_4)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} .$$

From this, we may write the linear operators $\mathcal{Z}'_i$;

$$\mathcal{Z}'_1 = \theta_1 + \theta_2 + \theta_3 + \theta_4\ ,\ \mathcal{Z}'_2 = 2\theta_2 + \theta_3 + \theta_4 + 1\ ,\ \mathcal{Z}'_3 = 3\theta_3 + \theta_4 + 1.$$

Since the period integral $\int_L \mathcal{U}(a)$ is annihilated by these operators, we have the following relations

$$\theta_2 = \theta_1 - 1\ ,\ \theta_3 = \theta_1 - 1\ ,\ \theta_4 = -3\theta_1 + 2\ , \tag{B.1}$$

when acting on $\int_L \mathcal{U}(a)$. Now consider the operator $\Box_l$ for $l = (1,1,1,-3)$. We have

$$a_1a_2a_3\Box_l \int_L \mathcal{U}(a) = \left(\theta_1\theta_2\theta_3 - \frac{a_1a_2a_3}{a_4^3}(\theta_4 - 2)(\theta_4 - 1)\theta_4\right) \int_L \mathcal{U}(a) = 0\ .$$

Then it is easy to observe the following factorization of the operator when we eliminate $\theta_{a_2}, \theta_{a_3}, \theta_{a_4}$ by (B.1);

$$(\theta_1 - 1)\left\{\theta_1(\theta_1 - 1) + 3\frac{a_1a_2a_3}{a_4^3}(3\theta_1 - 2)(3\theta_1 - 1)\right\} \int_C \mathcal{U}(a) = 0\ .$$

The inhomogeneous terms (i.e. constants) in (B.1) may be removed if we define $\Pi_L(a) := \frac{a_2a_3}{a_4^2} \int_L \mathcal{U}(a)$. Even in that case, the shape of the above factorized differential operator does not change. The resulting homogeneous linear operators imply that $\Pi_L(a) = \Pi_L(x)$ with $x = \frac{a_1a_2a_3}{a_4^3}$, and enable us to set $a_1 = a, a_2 = a_3 = a_4 = 1$ in which we have $\Pi_L(x) = \int_L \mathcal{U}(a)$ with $x = a$. Now we arrive at the second order differential equation,

$$\left\{\theta_a(\theta_a - 1) + 3a(3\theta_a - 2)(3\theta_a - 1)\right\} \Pi_L(a) = 0\ \ ,$$

by taking the irreducible part. This completes our derivation of (A.9).

Similar factorization property of the GKZ systems for primitive forms may be observed for G such that $\mathbf{C}^3/G$ has an isolated singularity. This factorization property is reminiscent the factorizations observed in [HKTY1] for the (extended) GKZ systems which appear in applications of mirror symmetry, although the irreducible part shows slightly different degenerations as we see above (cf. (A.10)).

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